Exercises for Numerical Methods I

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Sheet 3

due October 28, 2015

1. Compute the QR-decomposition of the matrix

$$A = \begin{bmatrix} 3 & 7 \\ 0 & 12 \\ 4 & 1 \end{bmatrix}.$$

2. Consider the function $f(x) = x^4$. Find the Lagrange-polynomial that interpolates the function f at the points $x_0 = -1,\ x_1 = 0$ and $x_2 = 2$.

3. Consider the sum

$$\sum_{k=0}^{n-1} \cos(j\, t_k) \sin(\hat{j}\, t_k),$$

where $t_k = k\frac{2\pi}{n}$ are the grid points. Using the formulas

$$\cos(j\, t_k) \sin(\hat{j}\, t_k) = \frac{1}{2} \operatorname{Im} \left\{ e^{i(j+\hat{j})t_k} - e^{i(j-\hat{j})t_k} \right\}$$

and

$$\sin(j\, t_k) \sin(\hat{j}\, t_k) = \frac{1}{2} \operatorname{Re} \left\{ e^{i(j-\hat{j})t_k} - e^{i(j+\hat{j})t_k} \right\},$$

calculate the above sums for all the possible values of $j, \hat{j} \in \left\{0, 1, ..., \frac{n-1}{2}\right\}$ following the same procedure as in the lecture notes.

4. Reminder: If p_i is a linear polynomial of the form

$$p_i(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i), \quad i = 0, ..., n$$

then $s(x) = p_i(x)$ for $x \in [x_i, x_{i+1}]$, where $s \in S_{1,\Delta}$ is the linear spline for the grid $\Delta = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of the interval $[a, b]$.

Let $f(x) = x^2,\ x \in [0, 3]$. Approximate the function f at the nodal points $x_i = i,\ i = 0, 1, 2, 3$ with a linear spline $s \in S_{1,\Delta}$.

5. Create a Matlab-Function that implements the linear spline interpolation for a given function $f : [a, b] \to \mathbb{R}$ and any grid Δ on the interval $[a, b]$. The algorithm should have the following structure:

   Input: Δ, i.e. the number of grid points.

Main body: Algorithm based on the polynomial of exercise 4.

Output: Graph of f and $s \in S_{1,\Delta}$.

Test your algorithm with the function

$$f(x) = \frac{1}{1+25x^2}, \quad x \in [-1,1],$$

and $n = 4, 8, 12$ equidistant grid points.

6. An efficient implementation in Matlab to calculate the Fourier coefficients a_k of the DFT

$$x_n = \sum_{k=0}^{N-1} a_k e^{ik\frac{2\pi}{N}n}, \quad n = 0, ..., N-1$$

is the function `fft`. To illustrate some of the properties of the DFT, consider different rectangular pulses of the form

$$x_n = \begin{cases} 1, & n \in [\frac{N-1}{2} - a, \frac{N-1}{2} + a] \\ 0, & \text{otherwise} \end{cases},$$

for a given positive integer a and N odd number. Additionally, consider the pulses,

$$x_{bn} = \begin{cases} 1, & n \in [\frac{N-1}{2} - a/b, \frac{N-1}{2} + a/b] \\ 0, & \text{otherwise} \end{cases},$$

and

$$x_n^{(b)} = \begin{cases} x_{n/b}, & n \in [\frac{N-1}{2} - ab : b : \frac{N-1}{2} + ab] \\ 0, & \text{otherwise} \end{cases},$$

related to decimation and time expansion, respectively. Using `fft` for $N = 31$, $a = 2$ and $b = 1, 2, 3$ present the Fourier transforms (using the graphs of a_k) for the different values of b.