Exercise Sheet 4

1. Consider the system $Ux = b$, where $U \in \mathbb{R}^{3\times 3}$ is an upper triangular matrix and $b \in \mathbb{R}^3$ an arbitrary vector. Show that the back substitution method for the solution of the above system is backward stable.

2. Use Gaussian elimination with partial pivoting to find a LU factorization of PA, where
$$A = \begin{bmatrix} 3 & 17 & 10 \\ 2 & 4 & -2 \\ 6 & 8 & -12 \end{bmatrix},$$
and P is a permutation matrix.

3. Let $A \in \mathbb{C}^{n\times n}$ and define $r_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|$. The *Gershgorin disks* D_i of A are the disks with midpoints a_{ii} and radii r_i. That is,
$$D_i = \{x \in \mathbb{C} : |x - a_{ii}| \leq r_i\}, \quad \text{for } i = 1, \ldots, n.$$
Gershgorin's theorem states that every eigenvalue of A lies in at least one Gershgorin disk. Use this theorem to show that *strictly diagonally dominant* matrices, i.e. matrices satisfying
$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad \text{for } i = 1, \ldots, n,$$
admit a LU factorization.

4. Let $A \in \mathbb{R}^{n\times n}$ be a tridiagonal matrix of the form
$$A = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_2 & a_2 & b_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & a_{n-1} & b_{n-1} \\ 0 & \cdots & 0 & c_n & a_n \end{bmatrix},$$
with a_k, b_k, $c_k \neq 0$, for all $k = 1, ..., n$.

If the matrix is strictly diagonally dominant, it admits a LU factorization where the matrices L and U have the special form

$$L = \begin{bmatrix} d_1 & 0 & \cdots & \cdots & 0 \\ c_2 & d_2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & c_{n-1} & d_{n-1} & 0 \\ 0 & \cdots & 0 & c_n & a_n \end{bmatrix}, \quad U = \begin{bmatrix} 1 & e_1 & 0 & \cdots & 0 \\ 0 & 1 & e_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & e_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

Write an algorithm for computing their entries d_k and e_k, $k = 1, ..., n$. How many operations are required?

5. Consider the linear system

$$Ax = b, \quad b \in \mathbb{R}^n,$$

for a tridiagonal strictly diagonally dominant matrix $A \in \mathbb{R}^{n \times n}$ as in Exercise 4. Write an algorithm for solving this linear system using the LU decomposition of A.

6. Implement in Matlab (or Octave) the algorithm of Exercises 4. Try to make your algorithm efficient by storing the coefficients d_k and e_k directly in A.

7. Implement in Matlab (or Octave) the algorithm of Exercises 5. Try to make your algorithm efficient by minimizing the use of `for`-loops.

8. Write a Matlab (or Octave) function that, for given regular matrix A and right hand side b, solves $Ax = b$ via Gaussian elimination with partial pivoting.