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Lecture Notes
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Chapter 1

Examples of ODEs and PDEs

1.1 Simple Examples of Ordinary Differential Equations in Applications

1.1.1 Movement of a Falling Body

We describe the movement of a vertically falling body. Then its position at time t is determined by its height $h(t)$.

Newton's second law of motion implies that the acceleration of the body, that is, the change of its speed, is proportional to the forces acting on the body. In addition, the proportionality constant equals the mass of the body. That is, the equation

$$F = m \cdot a$$

holds, where F denotes the forces, m the mass of the body, and a its acceleration.

Now, the acceleration is the change of the speed, which is itself the change of the position of the body. Therefore

$$F = m \cdot \ddot{h}(t) \ .$$

We still have to model the acting forces.

The main force is gravity, which, for small heights h, equals approximately $m \cdot g$, where $g \approx 9.81 \mathrm{m/s}^2$ is the gravitational acceleration at the earth's surface and m is again the mass of the body. The gravitation acting downwards, this implies the simple equation

$$m \cdot \ddot{h}(t) = -m \cdot g \ .$$

If either the body is very light or it is falling fast, it is necessary to take into account air friction as well, which will slow down the fall of the body. One possibility is to model air friction as a force proportional to the square of the body's velocity. Because friction always works against the current movement, the sign of the corresponding force will be opposite to the sign of $\dot{h}$. Thus we obtain the refined model

$$m \cdot \ddot{h}(t) = -c \operatorname{sgn}(()\dot{h}(t))\, \dot{h}(t)^2 - m \cdot g \,,$$

where c is some material constant describing the drag of the body.

In order to obtain a complete description of the movement of the body, we will need in addition a description of the state of the body at some initial time t_0, where we begin our considerations. More precisely, we will need its initial position h_0 and its initial velocity v_0. Then, assuming this model is correct, the movement of the body is completely described by the *differential equation*

$$\begin{aligned} m \cdot \ddot{h}(t) &= -c\,\mathrm{sgn}(()\dot{h}(t))\,\dot{h}(t)^2 - m \cdot g\,, \\ h(t_0) &= h_0\,, \\ \dot{h}(t_0) &= v_0\,. \end{aligned}$$

1.1.2 Population Dynamics

Now consider a simple model that describes the evolution of a population over some time period. That is, we know the population p_0 at some given time t_0, and we want to obtain an estimate $p(t)$ of the population at some future time $t > t_0$.

As a basic model, we assume that the *rate of change* of the population is given by some function $N(t,p)$ that depends only on the time and the size of the population. The time dependence can be used to model external influences on the population, for instance environmental changes, while size of the population influences the number of births and deaths, but can also be used to model overpopulation. Then the function p that describes the population solves the differential equation

$$\dot{p}(t) = N\big(t, p(t)\big)\,, \qquad\qquad p(t_0) = p_0\,.$$

One very simple model assumes that the number of births within a certain amount of time is proportional to the size of the population that is, the birth rate is constant. Then we obtain the equation

$$\dot{p}(t) = R\,p(t)$$

with $R > 0$ denoting the birth rate. Assuming, in addition, a constant death rate $S > 0$, the equation becomes

$$\dot{p}(t) = (R - S)\,p(t)\,.$$

Using the initial state $p(t_0) = p_0$, we obtain with this model the population dynamics

$$p(t) = p_0 e^{(R-S)(t-t_0)}\,.$$

That is, depending on the sign of $R - S$, either the population increases or decreases exponentially.

Now we try to introduce the effects of overpopulation into the model by assuming that the death rate depends on the size of the population. That is, instead of assuming a constant death rate $S > 0$, we assume that S is a function of p. The simplest model is to assume the death rate being proportional to p, setting

$$S(p) = \sigma\,p$$

for some constant $\sigma > 0$. Then we obtain the equation (the *logistic differential equation*)

$$\dot{p}(t) = \big(R - \sigma p(t)\big)\, p(t) \, . \tag{1.1}$$

In the following, we will compute the analytic solution of this equation. Before that, we study some qualitative properties of the solution. To that end note first that the derivative of p is positive if $R > \sigma p$ (and the population p is positive, which we tacitly assume), while it is negative if $R < \sigma p$. In other words, the population increases as long as $p < R/\sigma$, while it decreases for $p > R/\sigma$. In particular, this implies that the long term behavior of the population will be approximately stagnation at the value $p = R/\sigma$. Moreover, this behavior is independent of the initial value p_0, as long as it is strictly larger than zero.

In order to solve the logistic differential equation, we define

$$\rho := R/\sigma \, ,$$

and rewrite the equation as

$$\frac{1}{(\rho - p(t))\, p(t)} \frac{dp(t)}{dt} = \sigma \, .$$

Integrating (with an indefinite integral) both sides of this equation with respect to t, we obtain

$$\int \frac{1}{(\rho - p(t))\, p(t)} \frac{dp(t)}{dt} \, dt = \int \sigma \, dt + C$$

for some constant $C \in \mathbb{R}$. Now replace the integration variable on the left hand side of this equation by p. Because

$$\frac{dp}{dt} dt = dp \, ,$$

we obtain the equation

$$\int \frac{1}{(\rho - p)p} \, dp = \sigma t + C \, ;$$

Note, that now the integration is with respect to the variable p.

Now note that

$$\int \frac{1}{(\rho - p)p} \, dp = \frac{1}{\rho} \int \frac{1}{\rho - p} + \frac{1}{p} \, dp = \frac{1}{\rho}\big(-\ln|\rho - p| + \ln|p|\big) = \frac{1}{\rho} \ln |[| B] \frac{p}{\rho - p} \, .$$

Thus the function p satisfies the equation

$$\frac{1}{\rho} \ln |[| B] \frac{p}{\rho - p} = \sigma t + C \, .$$

Multiplying the equation with ρ and taking the exponential, it follows that

$$|[| B] \frac{p}{\rho - p} = e^{\rho\sigma t + \rho C} = e^{\rho\sigma t}\, e^{\rho C} \, ,$$

which is equivalent to

$$|[| B] \frac{\rho - p}{p} = e^{-\rho\sigma t}\, e^{-\rho C} \, ,$$

Now we define a new constant

$$D := \pm e^{-\rho C} \, .$$

Then this last equation reads as

$$\frac{\rho}{p} - 1 = De^{-\rho\sigma t} \, ,$$

which in turn implies that

$$p = \frac{\rho}{1 + De^{-\rho\sigma t}} \, .$$

This is the general form of a solution of the differential equation (1.1). The *specific solution* satisfying $p(t_0) = t_0$ can be obtain by choosing the constant D in a suitable manner.

1.2 Solution of ODEs

1.2.1 ODEs with Separable Variables

Definition 1.1. An ODE that can be brought into the form

$$f(y)\dot{y} = g(t) \, , \tag{1.2}$$

where the function $f\colon \mathbb{R} \to \mathbb{R}$ only depends on y and not on t, and the function $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ only depends on t and not on y, is called ordinary differential equation (of first order) with *separable variables.* ■

Now assume that we are given an ODE that we can bring in the form (1.2). Since

$$\dot{y} = \frac{dy}{dt} \, ,$$

we can *formally* multiply with dt and obtain the *formal* equation

$$f(y) \, dy = g(t) \, dt \, .$$

Now we can apply indefinite integrals to both sides and obtain the equation

$$\int f(y) \, dy = \int g(t) \, dt + C \, ,$$

where $C \in \mathbb{R}$ is some constant that appears due to the indefinite integration. Note, that the first integration is with respect to y and the right hand side reveals an integration with respect to t.

If it is possible to compute the integrals of f and g analytically, we obtain an equation the solution necessarily has to satisfy. If, in addition, it is possible to solve this equation for y, we indeed obtain an analytic (general) solution of the differential equation.

Example 1.2. Consider the ODE

$$(T^2 - t^2) \, \dot{y} + ty = 0 \, ,$$

where $T > 0$ is some given constant. This equation has separable variables, but in the form above they are not yet separated. In order to bring the equation in the form (1.2), we rewrite the equation as

$$\frac{\dot{y}}{y} = -\frac{t}{T^2 - t^2},$$

which is possible for $y \neq 0$ and $t \neq \pm T$. We rewrite this formally as

$$\frac{dy}{y} = -\frac{t}{T^2 - t^2} dt .$$

Now, integration of both sides of the equation leads to

$$\ln |y| = \frac{1}{2} \ln \left| T^2 - t^2 \right| + C .$$

Taking the exponential of the equation, we obtain

$$|y| = e^C \sqrt{|T^2 - t^2|} .$$

Replacing the constant $e^C > 0$ by the constant $D \in \mathbb{R}$ also encoding the sign of y, we get

$$y(t) = D\sqrt{|T^2 - t^2|} . \tag{1.3}$$

The constant $D \in \mathbb{R}$ still has to be determined using the initial condition $y(t_0) = y_0$. Inserting this condition into the general solution, we see that

$$y_0 = y(t_0) = D\sqrt{|T^2 - t_0^2|},$$

and therefore

$$D = \frac{y_0}{\sqrt{|T^2 - t_0^2|}} . \tag{1.4}$$

Note that we have assumed during the computation of the solution of the ODE that $y_0 \neq 0$ and $t \neq \pm T$. It can be easily seen, however, that the derivation above also covers the situation where $y_0 = 0$ and $t_0 \neq \pm T$. There, the constant function $y = 0$ is the unique solution of the ODE, at least until the time reaches one of the values $\pm T$.

The case $t_0 = \pm T$, however, is different. Then, if $y_0 = 0$, for *every* constant $D \in \mathbb{R}$ the function (1.3) satisfies the ODE and therefore is a solution. If, however, $y_0 \neq 0$, then the ODE has no solution at all—then the ODE and the initial conditions are inconsistent.

Finally, note that all the solutions are valid only locally; that is, there exists at least a time interval $[t_0, t_0 + \varepsilon)$ for some $\varepsilon > 0$ on which the solution exists and can be written as (1.3) with D given by (1.4). For general ODEs, this is all that can be said about the solution. In this special case, one can specify the length of the interval on which the solution looks like (1.3): If $t_0 > T$, then the formula (1.3) is valid on $[t_0, +\infty)$. If, however, $-T < t_0 < T$, then the solution is

$$y(t) = \begin{cases} D_1\sqrt{|T^2 - t^2|} & \text{if } t \in [t_0, T] , \\ D_2\sqrt{|T^2 - t^2|} & \text{if } t \in [T, +\infty) , \end{cases} \qquad \text{with} \begin{cases} D_1 = y_0/\sqrt{T^2 - t_0^2} , \\ D_2 \in \mathbb{R} \text{ arbitrary.} \end{cases}$$

In particular, the solution is only unique up to time T. Similarly, if $t_0 < -T$, then

$$y(t) = \begin{cases} D_1\sqrt{|T^2 - t^2|} & \text{if } t \in [t_0, -T]\,, \\ D_2\sqrt{|T^2 - t^2|} & \text{if } t \in [-T, T]\,, \\ D_3\sqrt{|T^2 - t^2|} & \text{if } t \in [T, +\infty)\,, \end{cases} \qquad \text{with } \begin{cases} D_1 = y_0/\sqrt{T^2 - t_0^2}\,, \\ D_2 \in \mathbb{R} \text{ arbitrary}, \\ D_3 \in \mathbb{R} \text{ arbitrary}. \end{cases}$$

■

1.2.2 Homogeneous ODEs

Definition 1.3. An ODE of the form

$$\dot{y} = f\Big(\frac{y}{t}\Big)\,, \tag{1.5}$$

with $f\colon \mathbb{R} \to \mathbb{R}$, is called of *homogeneous type*. ■

If we are given an ODE of homogeneous type, we can solve it by starting with the substitution

$$z(t) = \frac{y(t)}{t}\ .$$

For the right hand side of (1.5) we are left with the term $f(z)$. For the left hand side of (1.5) we use the product rule and obtain

$$\dot{y} = \frac{dy}{dt} = \frac{d(tz)}{dt} = z + t\frac{dz}{dt} = z + t\dot{z}\ .$$

Thus we have for the variable z the differential equation

$$z + t\dot{z} = f(z)\ .$$

Now it is easy to see that this ODE is of separable type: We can bring it in the form

$$\frac{\dot{z}}{f(z) - z} = \frac{1}{t}\ .$$

This ODE can now be solved by integration as in Section 1.2.1, and we obtain a solution $z(t)$. At the end, we obtain the solution y by $y(t) = tz(t)$.

Example 1.4. Consider the ODE

$$\dot{y} = \Big(\frac{y}{t}\Big)^2\ .$$

It is easy to see that this ODE is homogeneous with $f(y/t) = (y/t)^2$. Using the substitution $y = tz$ we obtain

$$z + t\dot{z} = z^2$$

and therefore

$$\frac{\dot{z}}{z^2 - z} = \frac{1}{t}\ .$$

Now, we follow Subsection 1.2.1 and reformulate this equation to

$$\frac{1}{z^2 - z}dz = \frac{1}{t}dt\ .$$

Integrating this equation, we obtain the indefinite integral equation

$$\int \frac{1}{z^2 - z} dz = \int \frac{1}{t} dt + C$$

or by calculating the integrals

$$\ln |[| B] \frac{z-1}{z} = \ln |t| + C \ .$$

Now, assuming that $t > 0$ and $z \geq 1$, which depends on the initial condition, we get

$$\frac{z-1}{z} = Dt$$

for some constant $D = \exp(C) \in \mathbb{R}^+$ depending on the initial value. Solving for z we obtain

$$z(t) = \frac{1}{1 - Dt}$$

(note that z is greater than 1) and, after substitution of $y = tz$

$$y(t) = \frac{t}{1 - Dt} \ .$$

■

1.3 Linear ODEs

Definition 1.5. An ODE that can be written as

$$\dot{y} + f(t)y = g(t)$$

for some functions $f \colon \mathbb{R} \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$ is called *linear ODE of first order*.■

Here, first order means that the highest derivative of the unknown function y that appears in the equation is the first one. Linear means that all the expressions are linear in the unknown y and its derivatives.

As in the case of linear algebraic equations, the linearity of an equation has some implications on the structure of its solutions. To that end we consider the *homogeneous* equation[1]

$$\dot{y} + f(t)y = 0 \ .$$

If we are given two solutions y_1 and y_2 of this equation (with possibly different initial conditions), then

$$\dot{y}_1 + f(t)y_1 = 0 \, ,$$
$$\dot{y}_2 + f(t)y_2 = 0 \ .$$

Consequently also

$$\frac{d}{dt}(y_1 + y_2) + f(t)(y_1 + y_2) = \dot{y}_1 + f(t)y_1 + \dot{y}_2 + f(t)y_2 = 0 \, ,$$

which shows that also $y_1 + y_2$ is a solution of the ODE. More general, if y_1 and y_2 solve the ODE and $c_1, c_2 \in \mathbb{R}$, then the linear combination $c_1 y_1 + c_2 y_2$ is also a solution.

[1] Homogeneous means that the right hand side of the equation is zero, that is, $g = 0$

In order to solve the (inhomogeneous) equation

$$\dot{y} + f(t)y = g(t) \tag{1.6}$$

we first observe that (1.6) is equivalent to

$$h(t)\dot{y} + h(t)f(t)y = h(t)g(t)\,, \tag{1.7}$$

at least, if $h\colon \mathbb{R} \to \mathbb{R}$ is a function that is different from zero.

Now the idea is to choose the function h in such a way that the left hand side of (1.7) is itself a derivative. More precisely, we try to find $h\colon \mathbb{R} \to \mathbb{R}$ such that

$$h(t)\dot{y} + h(t)f(t)y = \frac{d}{dt}(hy) = \dot{h}y + h\dot{y}\,. \tag{1.8}$$

If (1.8) holds, then the equation (1.7) reads as follows,

$$\frac{d}{dt}(hy) = h(t)g(t)\,,$$

which after integration becomes:

$$h(t)y(t) = \int^t g(s)h(s)\,ds + C\,. \tag{1.9}$$

For this reason, a function h satisfying (1.8) is called an *integrating factor* for the ODE (1.6).

Thus, (1.8) is satisfied, if h satisfies

$$h(t)f(t)y = \dot{h}(t)y\,.$$

Dividing this equation by y, we see that h has to satisfy the ODE

$$\dot{h} = f(t)h\,.$$

This ODE can be solved by separation of the variables, and we obtain the integrating factor

$$h(t) = D\exp\Big(\int^t f(s)\,ds\Big)\,.$$

Inserting this integrating factor in (1.9), we obtain

$$y(t) = \frac{\int^t \Big[g(s)D\exp\Big(\int^s f(r)\,dr\Big)\Big]ds + C}{D\exp\Big(\int^t f(s)\,ds\Big)}\,,$$

or, setting $\tilde{C} := C/D$,

$$y(t) = \frac{\int^t \Big[g(s)\exp\Big(\int^s f(r)\,dr\Big)\Big]ds + \tilde{C}}{\exp\Big(\int^t f(s)\,ds\Big)}\,.$$

1.4 Simple Examples of Partial Differential Equations

Definition 1.6. A partial differential equation (PDE) is an equation for more than two different derivatives of a function $u(x_1, x_2, \dots, x_n)$ on a domain $\Omega \subseteq \mathbb{R}^n$. ■

Example 1.7. For instance

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = x^2 yu \text{ or } yu_{xx} + u_y = x^2 yu\,, \tag{1.10}$$

which actually means

$$y\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial u}{\partial y}(x,y) = x^2 yu(x,y) \text{ for } (x,y) \in \Omega\,. \tag{1.11}$$

This is a PDE for a function $u(x,y)$ in two variables. ■

We use the notation that

$$u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y},$$

and

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial \frac{\partial u}{\partial x}}{\partial x}, u_{xy} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial \frac{\partial u}{\partial x}}{\partial y} = \frac{\partial \frac{\partial u}{\partial y}}{\partial x} = u_{yx}\,.$$

- The variables x, y are called *independent* variables.
- u is called *dependent* variable.

The *order* of the differential equation is the order of the highest derivative of the dependent variables in the differential equation. The PDE (1.11) is a differential of second order. The differential equation

$$xu_x u_{xxy} + u_x^4 = 0$$

is of third order.

Most PDEs with relevance in practice are of first or second order.

Example 1.8.
1. The electrostatic potential $u(x,y,z)$ which is determined by a charge density $\rho(x,y,z)$ satisfies the *Poisson equation*

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 4\pi\rho\,.$$

   Δ denotes the *Laplace* operator (in space dimension three).

2. The *wave equation* is the PDE

$$\frac{1}{c^2}u_{tt} = \Delta u\,.$$

   In air, $u(x,y,z,t)$ denotes the density of air at a location (x,y,z) at time t. c denotes the sound speed.

3. *Heat* or *diffusion* equation:

$$u_t = \alpha \Delta u \,,$$

with some $\alpha > 0$.

4. The velocity $\vec{v}(\vec{x},t) = (v_1, v_2, v_3)(\vec{x},t)$ and the pressure $p(\vec{x},t)$ of an incompressible fluid as a function of space $\vec{x} = (x,y,z)$ and time t satisfies the *Navier-Stokes-equations*

$$\vec{v}_t + (\vec{v}\cdot\nabla)\vec{v} = \nu\Delta\vec{v} + \nabla p \,, \nabla\cdot\vec{v} = 0 \,,$$

for some constant ν. In the above equation

$$\nabla\cdot\vec{v} := (v_1)_x + (v_2)_y + (v_3)_z$$

denotes the *divergence*.

$$(\vec{v}\cdot\nabla)\vec{v} = \begin{pmatrix} v_1(v_1)_x + v_2(v_1)_y + v_3(v_1)_z \\ v_1(v_2)_x + v_2(v_2)_y + v_3(v_2)_z \\ v_1(v_3)_x + v_2(v_3)_y + v_3(v_3)_z \end{pmatrix} .$$

This is a system of four equations in four unknowns. ■

Definition 1.9. A PDE is called *linear* if u and its derivative only appear *linearly*. More precisely a linear PDE has the form

$$Lu = b \,,$$

where L is a differential operator and b is a given function. ■

Example 1.10. Equation (1.10) is linear with

$$L = y\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} - x^2 y \,, \qquad b = 0 \,.$$

The Poisson equation is linear with $L = \Delta$ and $b = 4\pi\rho$. The wave equation and the heat equation are linear, respectively. The Navier-Stokes equation is nonlinear. Another, frequently used nonlinear PDE is the *Burger's equation*

$$u_x + u u_y = 0 \,.$$

1.4.1 Conservation Principles

Differential equations are frequently derived from conservation of physical quantities like mass, energy, temperature and so on. To illustrate this we consider the temperature distribution in a homogeneous, non insulating slab of length L. We denote now by $u(x,t)$ the temperature in a point $x \in [0,L]$ at time $t \geq 0$.

We are modeling the following principles:

1. Conservation of energy: The timely variation of thermal energy in every interval $[a,b] \subseteq [0,L]$ is equal to the heat flux across a and b.

2. The energy density (energy per length's unit) is $\rho c u$. Thereby ρ denotes the density, c is the specific heat. Both ρ and c are assumed to be constant here.

3. Fourier's law: The heat flux is proportional to the gradient of the temperature (heat is fluctuating from warm to cold), and the proportionality constant $k > 0$ is called heat conductivity. That is, we have

$$\frac{d}{dt}\int_a^b \rho c u(x,t)\,dx = -k\frac{\partial u}{\partial x}(a,t) + k\frac{\partial u}{\partial x}(b,t)\,. \tag{1.12}$$

The left hand side is the variation of the total energy in the slab. The right hand side is the energy, which migrates in and out of the slab per time unit. Equation (1.12) should hold for all intervals $[a,b]$. Thus by the fundamental theorem of integration we get from (1.12)

$$\int_a^b \left(\rho c\frac{\partial u}{\partial t}(x,t)\,dx - k\frac{\partial^2 u}{\partial x^2}(x,t)\right)\,dx = 0\,.$$

Because this holds for arbitrary intervals $[a, a+\varepsilon]$ we see that

$$\begin{aligned} 0 &= \frac{1}{\varepsilon}\int_a^{a+\varepsilon} \left(\rho c\frac{\partial u}{\partial t}(x,t)\,dx - k\frac{\partial^2 u}{\partial x^2}(x,t)\right)\,dx \\ &\sim \left(\rho c\frac{\partial u}{\partial t}(a,t) - k\frac{\partial^2 u}{\partial x^2}(a,t)\right) \end{aligned}$$

This should hold for all a, which gives that

$$0 = \frac{\partial u}{\partial t}(a,t) - \underbrace{\frac{k}{\rho c}}_{:=\alpha}\frac{\partial^2 u}{\partial x^2}(a,t)\,.$$

This is the heat equation in $\mathbb{R} \times \mathbb{R}_0^+$.

In $\mathbb{R}^n$ the derivation from Fourier's law is similar and results in

$$0 = \frac{\partial u}{\partial t}(a,t) - \underbrace{\frac{k}{\rho c}}_{:=\alpha}\Delta u(a,t)\,,$$

for all $a \in \mathbb{R}^3$. Here

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

denotes the Laplace operator.

Chapter 2

Numerical Solution of Partial Differential Equations as Systems of ODEs

We explain at an example how we can reformulate a partial differential equation as a system of ordinary differential equations. In this way we can apply all numerical methods for solving ordinary equations to solve this partial differential equatios, such as Euler methods, Runge-Kutta methods, Adams-Bashforth, to name but a few.

Example 2.1. Let $u(x,t)$, $-1 \leq x \leq 1$, be the temperature distribution at time t in a slab of length $l = 2$. Assuming constant conductivity $\sigma = 1$, u satisfies the *heat conduction* equation:

$$u_t = \sigma u_{xx} = u_{xx}\,, \quad -1 < x < 1\,, 0 < t < T\,. \tag{2.1}$$

This is now a *partial differential equation* because it depends on derivatives of two variables x, t. By discretization of the x variable we can transform the partial differential equation in a system of ordinary differential equations.

Let $v : [-1, 1] \to \mathbb{R}$ be an arbitrary function satisfying $v(-1) = v(1) = 0$, then we get by integration by parts

$$\int_{-1}^{1} u_t(t,x)v(x)\;dx = \int_{-1}^{1} u_{xx}(t,x)v(x)\;dx = -\int_{-1}^{1} u_x(t,x)v_x(x)\;dx\,. \tag{2.2}$$

Assume that the temperatures $u(-1,t) := u_0(t)$ and $u(1,t) := u_1(t)$ are measured, then, for every $t > 0$, $u(t,x)$ can be approximated by a linear spline in space over the grid $\Delta = \{-1 = x_0 < x_1 < ... < x_n = 1\}$, that is

$$u(t,x) = \sum_{i=0}^{n} y_i(t)\Lambda_i(x)\,, \tag{2.3}$$

where Λ_i is a linear hat function with peak at x_i. Taking into account the boundary conditions we see that $y_0 = u_0(t)$ and $y_n = u_1(t)$. All other functions y_i are unknown.

Inserting (2.3) in (2.2) we get a system of differential equations for $y_1, ..., y_{n-1}$:

$$\sum_{i=0}^{n} y_i'(t) \int_{-1}^{1} \Lambda_i(x) v(x)\, dx = -\sum_{i=0}^{n} y_i(t) \int_{-1}^{1} \Lambda_i'(x) v_x(x)\, dx \ ,$$

where we choose $v(x) \in \{\Lambda_j(x) : j = 1, ..., n-1\}$ - this means that v is a hat function, which satisfies homogenous boundary conditions.

Denote by

$$G := [\langle \Lambda_i, \Lambda_j \rangle]_{1 \le i,j \le n-1} = \frac{h}{6} \begin{bmatrix} 4 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 4 & 1 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 1 & 4 & 1 \\ \vdots & . & . & 0 & 1 & 4 \end{bmatrix}$$

and

$$A := [\langle \Lambda_{ix}, \Lambda_{jx} \rangle]_{1 \le i,j \le n-1} = h \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & -1 & 2 & -1 \\ \vdots & \ddots & \ddots & 0 & -1 & 2 \end{bmatrix}$$

we get a compact description of the system

$$Gy'(t) + Ay(t) = b(t)\,, \tag{2.4}$$

where b is an appropriate vector, which depends on u_0 and u_1.

To completely specify the system (2.4) we need initial values for $y_1, ..., y_{n-1}$, which are typically determined from interpolation of the initial temperature $u(0, x)$. ∎

Chapter 3

Classification of Linear Partial Differential Equations

For the sake of simplicity of presentation we restrict attention to linear partial differential equations of second order in two variables. Such an equation for a function $u = u(x, y)$ reads as follows:

$$Au_{xx} + 2B_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0 \ . \qquad (3.1)$$

Here $A = A(x, y), \ldots, G = G(x, y)$ are again functions.

Definition 3.1. A PDE (3.1) is called

- *elliptic* if $AC - B^2 > 0$,
- *parabolic* if $AC - B^2 = 0$, and
- *hyperbolic* if $AC - B^2 < 0$. ■

Example 3.2.
The wave equation

$$\frac{1}{c^2}u_{xx} - u_{yy} = 0$$

(note we changed the notation from t to y) is of the from (3.1) with

$$A = c^{-2}, B = 0, C = -1 \ .$$

Because

$$AC - B^2 = -c^{-2} < 0 \,,$$

the equation is hyperbolic.
The Laplace equation

$$u_{xx} + u_{yy} = 0$$

is of the from (3.1) with

$$A = 1, B = 0, C = 1 \ .$$

Because

$$AC - B^2 = 1 \geq 0\,,$$

the equation is elliptic.

For the heat equation

$$u_x - u_{yy} = 0$$

(note we changed the notation from t to x and x to y) is of the from (3.1) with

$$A = 0, B = 0, C = -1\,.$$

Because

$$AC - B^2 = 0\,,$$

the equation is parabolic. ■

Remark 3.3. 1. For the classification on the *main symbol*

$$Au_{xx} + 2Bu_{xy} + Cu_{yy}$$

is relevant. These are the terms of the differential equation of highest order (in our case this is 2).

2. $AC - B^2$ is the determinant of the symmetric matrix

$$M = \begin{pmatrix} A & B \\ B & C \end{pmatrix}\,.$$

Denoting by $m_{11} = A$, $m_{12} = m_{21} = B$, $m_{22} = C$ and $x_1 = x$, $x_2 = y$ the main symbol reads as follows

$$\sum_{i,j=1}^{2} m_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}\,.$$

3. If the coefficients A, B, C are not constant, but functions which depend on x and y in a non-trivial manner, then the type of the partial differential equation can be different at various points (x, y).

For instance the differential equation

$$xu_{xx} + u_{yy} = 0$$

is elliptic for $x > 0$, parabolic for $x = 0$, and hyperbolic for $x < 0$.

4. The terminology elliptic, parabolic and hyperbolic is motivated from conic sections. A curve $(X, Y(X))$, which satisfies the equation

$$AX^2 + 2BXY + CY^2 + DX + EY + F = 0 \qquad \text{(with constant coefficients)}$$

is either an ellipsis, parabola, or an hyperbola, depending on the sign of $AC - B^2$.

A different terminology is used in linear algebra, where *quadratic forms* are investigated:

$$Q(X, Y) = AX^2 + 2BXY + CY^2\,.$$

We assume that one of the coefficients A, B, C is not identical 0.

- Q is called positive (negative) definite, if $Q(X,Y) > 0$ ($Q(X,Y) < 0$) for all $(X,Y) \neq (0,0)$.
- Q is called positive (negative) semi-definite, if $Q(X,Y) \geq 0$ ($Q(X,Y) \leq 0$) for all $(X,Y) \neq (0,0)$.
- If Q is not semi definite, then it is called *indefinite*.

It is a theorem of Linear Algebra that

- Q is positive or negative definite if and only if $AC - B^2 > 0$;
- It is indefinite iff $AC - B^2 < 0$;
- It is semi-definite iff $AC - B^2 = 0$.

By a change of coordinates every quadratic form can be transformed into

- $X^2 + Y^2$ if $AC - B^2 > 0$,
- $X^2 - Y^2$ if $AC - B^2 < 0$,
- X^2 if $AC - B^2 = 0$.

For instance,

$$\begin{aligned} Q(X,Y) &= X^2 + 4XY + Y^2 \\ &= (X+2Y)^2 - 4Y^2 + Y^2 \\ &= (X+2Y)^2 - 3Y^2 \\ &= (X')^2 - (Y')^2 , \end{aligned}$$

∎

where

$$X' = X + 2Y \text{ and } Y' = \sqrt{3}Y .$$

Remark 3.4. An important property of the classification, Definition 3.1, is that it is invariant under coordinate transformations: A coordinate transformation does not change the type of the differential equation.

We consider the change of coordinates:

$$\begin{aligned} L_1(x,y) &= ax + by , \\ L_2(x,y) &= cx + dy . \end{aligned}$$

This shows that

$$\begin{aligned} \frac{\partial(u \circ L)}{\partial x}(x,y) &= a\partial_1 u(L(x,y)) + c\partial_2 u(L(x,y)) , \\ \frac{\partial(u \circ L)}{\partial y}(x,y) &= b\partial_1 u(L(x,y)) + d\partial_2 u(L(x,y)) . \end{aligned}$$

The notation is ugly, however, it should make aware that on the right hand side we differentiate with respect to the first component and not x variable.

Thus

$$\begin{aligned} \frac{\partial^2(u \circ L)}{\partial^2 x}(x,y) &= a^2\partial_1^2 u(L(x,y)) + 2ac\partial_{12}^2 u(L(x,y)) + c^2\partial_2^2 u(L(x,y)) , \\ \frac{\partial^2(u \circ L)}{\partial x \partial y}(x,y) &= ab\partial_1^2 u(L(x,y)) + (ad+bc)\partial_{12}^2 u(L(x,y)) + cd\partial_2^2 u(L(x,y)) , \\ \frac{\partial^2(u \circ L)}{\partial^2 y}(x,y) &= b^2\partial_1^2 u(L(x,y)) + 2bd\partial_{12}^2 u(L(x,y)) + d^2\partial_2^2 u(L(x,y)) . \end{aligned}$$

This can be written now into compact matrix form:

$$\underbrace{\begin{pmatrix} \frac{\partial^2(u\circ L)}{\partial^2 x}(x,y) & \frac{\partial^2(u\circ L)}{\partial x \partial y}(x,y) \\ \frac{\partial^2(u\circ L)}{\partial x \partial y}(x,y) & \frac{\partial^2(u\circ L)}{\partial^2 y}(x,y) \end{pmatrix}}_{=:C}$$

$$= A \underbrace{\begin{pmatrix} \partial_1^2 u(L(x,y)) & \partial_{12}^2 u(L(x,y)) \\ \partial_{12}^2 u(L(x,y)) & \partial_2^2 u(L(x,y)) \end{pmatrix}}_{=:U''} A^T$$

with

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} .$$

Thus we have

$$(\det(A))^2 \det(U'') = \det(C) ,$$

and thus the determinants of C and A have equal signs. This shows that the type does not change by linear transformations. ■

3.1 Characteristics

The following considerations make evident the role of the different types of differential equations.

Exemplary, first, we consider an ordinary differential equation:

$$u''(x) = F(x, u(x), u'(x)) .$$

Suppose that $u(x_0)$ and $u'(x_0)$ are known in a point x_0, then (compare the Euler method)

$$\begin{aligned} u(x_0 + \Delta x) &\approx u(x_0) + u'(x_0)\Delta x , \\ u'(x_0 + \Delta x) &\approx u'(x_0) + u''(x_0)\Delta x \\ &= u'(x_0) + F(x_0, u(x_0), u'(x_0))\Delta x . \end{aligned}$$

The equations get exact for $\Delta x \to 0$. That is, the solution of the differential equation can be determined approximately from the values at x_0. In the following we apply this idea to partial differential equations: We use the linearization:

$$u(x_0 + \Delta x, y_0 + \Delta y) \approx u(x_0, y_0) + u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y .$$

But we need also approximations for higher order derivatives which can be derived by approximation of u_x:

$$\begin{aligned} u_x(x_0 + \Delta x, y_0 + \Delta y) &\approx u_x(x_0, y_0) + u_{xx}(x_0, y_0)\Delta x + u_{xy}(x_0, y_0)\Delta y , \\ u_y(x_0 + \Delta x, y_0 + \Delta y) &\approx u_y(x_0, y_0) + u_{yx}(x_0, y_0)\Delta x + u_{yy}(x_0, y_0)\Delta y . \end{aligned}$$

This means that if you know the function value u at (x_0, y_0) and derivatives of up to second order at (x_0, y_0), then one knows also u and it's first derivatives in a neighborhood, that is at $(x_0 + \Delta x, y_0 + \Delta y)$.

This idea is generalized now to the PDE (3.1)

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = -Du_x - Eu_y - Fu - G ,$$

We now assume that we know u_x, u_y and u on a curve (not just on (x_0, y_0)). Then we also know the tangential derivatives in tangential direction $\vec{v}$ on the curve:

$$D_{\vec{v}} u_x = \lim_{t \to 0} \frac{u_x(x_0 + t\vec{v}) - u_x(x_0)}{t} \text{ and } D_{\vec{v}} u_y = \lim_{t \to 0} \frac{u_y(x_0 + t\vec{v}) - u_y(x_0)}{t} \, .$$

The tangential derivative can be expressed as

$$D_{\vec{v}} u_x = \nabla u_x \cdot \vec{v} \text{ and } D_{\vec{v}} u_y = \nabla u_y \cdot \vec{v} \, .$$

And therefore

$$\begin{aligned} v_1 u_{xx} + v_2 u_{xy} \qquad &= D_{\vec{v}} u_x \, , \\ v_1 u_{xy} + \qquad v_2 u_{yy} &= D_{\vec{v}} u_y \, . \end{aligned}$$

In summary, we know that for a given curve with tangential direction $\vec{v}$ and given u_x, u_y and u on a piece of the curve the second order derivatives from the system

$$\begin{aligned} v_1 u_{xx} + v_2 u_{xy} \qquad\qquad &= D_{\vec{v}} u_x \, , \\ v_1 u_{xy} + \quad v_2 u_{yy} &= D_{\vec{v}} u_y \, , \\ A u_{xx} + 2B u_{xy} + \quad C u_{yy} &= -D u_x - E u_y - F u - G \, . \end{aligned} \tag{3.2}$$

This linear system has a unique solution if

$$\det \begin{pmatrix} A & 2B & C \\ v_1 & v_2 & 0 \\ 0 & v_1 & v_2 \end{pmatrix} = A v_2^2 - 2B v_1 v_2 + C v_1^2 =: Q(v_2, -v_1) \tag{3.3}$$

does not vanish. This means that we can solve the PDE (locally around a point on curve) exactly.

Now, we reformulate the function Q:

$$\begin{aligned} Q(v_2, -v_1) &= A v_2^2 - 2B v_1 v_2 + C v_1^2 \\ &= A \left(v_2^2 - 2 \frac{B}{A} v_1 v_2 + \frac{C}{A} v_1^2 \right) \\ &= A \left(\underbrace{(v_2 - \frac{B}{A} v_1)}_{=: \tilde{v}_2}{}^2 + \left(\frac{B^2}{A^2} + \frac{C}{A} \right) v_1^2 \right) \, . \end{aligned}$$

Depending on the sign of $AC - B^2$ we have different scenarios:

Elliptic case: In this case $AC > B^2$, which in particular implies that $A \neq 0$. Then $Q(v_2, -v_1)$ implies that $\tilde{v}_2 = v_1 = 0$, and consequently $v_1 = v_2 = 0$.

Hyperbolic case: In the case $AC < B^2$, we can have the case $A = 0$ in which every $\vec{v}$ is a solution of $Q(v_2, -v_1) = 0$. If $A \neq 0$, then $\vec{v}$ is a solution of $Q(v_2, -v_1) = 0$ if

$$\tilde{v}_2 = \left(\frac{B^2}{A^2} + \frac{C}{A} \right) v_1 \, ,$$

which is solveable for a one-dimensional subspace. Thus it has non-trivial solutions, and thus the system (3.2) has nontrivial solutions too. That means that the solution of the PDE is not uniquely determined by the values u, u_x, u_y on the curve with tangent vector $\vec{v}$. Curves with such a property are called *characteristics*.

Because $\vec{n} = (v_2, -v_1)$ is the normal vector to the tangent we get the following definition of characteristics:

Definition 3.5. Let $AC - B^2 < 0$. A curve in the xy-plane is called *characteristics* of the hyperbolic PDE (3.1) if the normal vector $\vec{n} = (n_1, n_2)$ satisfies in every point the equation

$$An_1^2 + 2Bn_1n_2 + Cn_2^2 = 0 \ .$$

Example 3.6. The characteristics of the equation

$$u_{xx} - u_{yy} = 0 \,, \qquad (A = 0, C = 1, B = 0)$$

are the curves, where the normal vector satisfies

$$n_1^2 - n_2^2 = 0 \ .$$

That is $n_1 = \pm n_2$. That are the lines $x \pm y = \text{const}$.

If we identify y with time t, this is the standard wave equation. The characteristics $x \pm t = \text{const}$ are the lines, where the waves propagate:

If we consider waves, which move to the right, then this waves are given by

$$u(x,t) = F(x - t) \ .$$

Note, that on the characteristics the value is constant. The ones which move to the left are

$$u(x,t) = G(x + t) \ .$$

An in general the waves are of the form

$$u(x,t) = F(x - t) + G(x + t) \ .$$

This example makes clear the roles of characteristics. For an initial value problem with an initial values on a characteristics the equation is not unique solvable.

Chapter 4

Boundary Value Problems

For motivating purposes we study first boundary value problems for ordinary differential equations at hand of a simple test example:

$$\begin{aligned} L[u] = -u'' + bu' + cu = f \text{ in } (0,1)\,, \\ u(0) = u(1) = 0\,. \end{aligned} \tag{4.1}$$

it can be shown that this differential equation has a unique solution provided that

$$c(x) \geq 0\,, \qquad \forall x \in (0,1)\,.$$

This will always be assumed in the following.

Here, for the numerical solution, we consider *finite difference methods* (FDM). Later, alternatively we will investigate later *finite element methods* (FEM).

For the simplicity of presentation we consider an äquidistant grid

$$\Delta_h = \{x_i = ih : i = 1, \dots, n-1, h = 1/n\} \subseteq (0,1)\,. \tag{4.2}$$

We denote by

$$\vec{u} = (u(x_1), \dots, u(x_n))^t \in \mathbb{R}^{n-1} \tag{4.3}$$

the vector of the exact solution u of (4.1) on the grid Δ_h (4.2). In addition, we assume boundary conditions

$$0 = u(x_0) = u(x_n) = 0\,.$$

For the numerical solution we look for an approximating vector

$$\vec{u}_h = (u_1, \dots, u_n)^t \in \mathbb{R}^{n-1}\,. \tag{4.4}$$

For this purpose we discretize L from (4.1) by approximating the derivatives of u at the positions $x = x_i$ via *difference quotients*. Thereby we have several alternatives:

- *One-sided forward-difference operator:*

$$D_h^+[u](x) = \frac{u(x+h) - u(x)}{h} \sim u'(x)\,.$$

- *One-sided backward-difference operator:*

$$D_h^-[u](x) = \frac{u(x) - u(x-h)}{h} \sim u'(x) \ .$$

- *Central difference quotient:*

$$D_h[u](x) = \frac{u(x+h) - u(x-h)}{2h} \sim u'(x) \ . \tag{4.5}$$

Moreover, the second derivative can be approximated by a central difference quotient

$$D_h^2[u](x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \sim u''(x) \ . \tag{4.6}$$

Example 4.1. We study a simple situation of (4.1) with $b, c \equiv 0$, that is $-u'' = f$. We approximate u'' by $D_h^2[u]$ at the nodal points Δ_h. Taking into account the Dirichlet boundary conditions $u(x_0) = u(x_n) = 0$ we get the discretized equation:

$$\underbrace{\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \end{bmatrix}}_{=:\vec{f}} = - \begin{bmatrix} u''(x_1) \\ u''(x_2) \\ \vdots \\ u''(x_{n-1}) \end{bmatrix} \sim \underbrace{h^{-2} \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}}_{=:L_h} \underbrace{\begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{n-1}) \end{bmatrix}}_{\vec{u}} .$$

Because $\vec{u}$ should be approximated by $\vec{u}_h$, we will use the following linear equation to determine $\vec{u}_h$:

$$L_h \vec{u}_h = \vec{f} \ . \tag{4.7}$$

The Eigenvalues of L_h are $4h^{-2}\sin^2(kh\pi/2)4$, $k = 1, \ldots, n-1$. The function $\operatorname{sinc}(x) := \frac{\sin(x)}{x}$ is monotonically decreasing in $[0, \pi/2]$ such that

$$\operatorname{sinc}(x) \geq \operatorname{sinc}\left(\frac{\pi}{2}\right) = \frac{2}{\pi} , \quad \forall x \in [0, \pi/2] ,$$

which implies that:

$$\|L_h^{-1}\|_2 = \frac{1}{\lambda_{\min}(L_h)} = \max_{1 \leq k \leq n-1} \frac{h^2}{4\sin^2(kh\pi/2)} \leq \frac{1}{4} \ .$$

Consequently,

$$\begin{aligned} \|\vec{u} - \vec{u}_h\|_2 &= \|L_h^{-1}(L_h\vec{u} - \vec{f})\|_2 \\ &\leq \|L_h^{-1}\|_2 \|L_h\vec{u} - \vec{f}\|_2 \\ &\leq \frac{1}{4}\|L_h\vec{u} - \vec{f}\|_2 \ . \end{aligned} \tag{4.8}$$

If $\|L_h\vec{u} - \vec{f}\|_2$ converges to 0 for $h \to 0$, then L_h is called *consistent.* If there exists an estimate of the form (4.8), then *consistency* implies *stability.* ■

In the following we determine error estimates for difference quotients:

Lemma 4.2. *Let $u \in C^2[0,1]$ and $x \in [h, 1-h]$. Then, for one-sided difference quotients we have the estimate*

$$\left|D_h^{\pm}[u](x) - u'(x)\right| \leq \frac{1}{2}\|u''\|_\infty h \,.$$

For a central difference quotient and $u \in C^3[0,1]$ we even have:

$$|D_h[u](x) - u'(x)| \leq \frac{1}{6}\|u'''\|_\infty h^2 \,.$$

For D_h^2 we have: Let $u \in C^4[0,1]$ and $x \in [h, 1-h]$, then:

$$\left|D_h^2[u](x) - u''(x)\right| \leq \frac{1}{12}\|u''''\|_\infty h^2 \,. \tag{4.9}$$

Proof. We prove exemplary the assertion for the central difference quotient. Let $u \in C^3[0,1]$, then it follows from Taylor expansion around $x \in (0,1)$:

$$\begin{aligned} u(x+h) &= u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \frac{1}{6}h^3u'''(\zeta_+)\,, \\ u(x-h) &= u(x) - hu'(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3u'''(\zeta_-)\,, \end{aligned}$$

for some $\zeta_\pm$ satisfying $x - h < \zeta_- < x < \zeta_+ < x + h$. Therefore

$$u(x+h) - u(x-h) = 2hu'(x) + \frac{1}{6}h^3(u'''(\zeta_+) + u'''(\zeta_-))\,,$$

and thus

$$\left|\frac{u(x+h) - u(x-h)}{2h} - u'(x)\right| \leq \frac{1}{6}h^2 \sup\left\{|u'''(\zeta)| : \zeta \in [0,1]\right\}\,,$$

which gives the assertion. □

Example 4.3. Applied to Example 4.1 we find that, provided the solution of the differential equation is 4× continuously differentiable, that

$$\|\underbrace{L_h}_{=D_h^2}\vec{u} - \vec{f}\|_\infty \leq \frac{1}{12}\|u''''\|_\infty h^2 = \frac{1}{12}\|f''\|_\infty h^2 \,.$$

In the following we discretize the operator L defined in (4.1). We use the discretization $D_h^2[u]$ for approximating u''. Moreover, the first derivative is approximated by either one of the difference quotients $D_h^+[u]$, $D_h^-[u]$, $D_h[u]$. Using different difference quotients gives different diagonal matrices:

$$L_h = h^{-2}\begin{bmatrix} d_1 & s_1 & & 0 \\ r_2 & d_2 & \ddots & \\ & \ddots & \ddots & s_{n-2} \\ 0 & & r_{n-1} & d_{n-1} \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}\,, \tag{4.10}$$

where for

- D_h^+:

$$\begin{aligned} d_i &= 2 - hb(x_i) + h^2c(x_i)\,, \\ r_i &= -1\,, \\ s_i &= -1 + hb(x_i)\,, \end{aligned} \tag{4.11}$$

- D_h^-:

$$\begin{aligned} d_i &= 2 + hb(x_i) + h^2 c(x_i)\,, \\ r_i &= -1 - hb(x_i)\,, \\ s_i &= -1\,, \end{aligned} \tag{4.12}$$

- D_h:

$$\begin{aligned} d_i &= 2 + h^2 c(x_i)\,, \\ r_i &= -1 - hb(x_i)/2\,, \\ s_i &= -1 + hb(x_i)/2\,. \end{aligned} \tag{4.13}$$

The approximate solution is determined as the solution of the linear system (4.7). ■

Definition 4.4. A difference method has *order of consistence* q if

$$\|L_h\vec{u} - \vec{f}\|_\infty = \max|(L_h\vec{u})_i - f_i| \le Ch^q\,.$$

Note, that in this definition $\vec{u}$ is the vector of the solution of the infinite dimensional problem at the nodal points. ■

Theorem 4.5. *Let the solution of the boundary value problem* (4.1) *be* $4\times$ *continuously differentiable (which is for instance the case if* b, c, f *are* $2\times$ *continuously differentiable). Then the difference method* (4.7) *has the order of consistency* q*:*

- $q = 2$*, if the central difference quotient* D_h *is used for approximating* u'*;*
- $q = 1$*, if forward or backward difference quotients* $D_h^\pm$ *are used for approximating* u'*.*