Exercise Sheet 1

1. Given the LPP:

$$\begin{aligned} \max \quad & c^T x \\ s.t. \quad & Ax \leq b \\ & x \geq 0, \end{aligned}$$

   where $c \in \mathbb{R}^n$, $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $b \geq 0$ and $a_{ij} > 0$ for at least one $i \in \{1, ..., m\}$ and all $j = 1, ..., n$. Show that the above LPP admits an optimal solution.

2. Prove Theorem 2.3.4 from the lecture notes.

3. Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Then, show that:

   (a) If $\{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset$ then $\{z \in \mathbb{R}^m : A^T z = 0 \text{ and } b^T z = 1\} = \emptyset$.

   (b) If $\{x \in \mathbb{R}^n : Ax = b \text{ and } x \geq 0\} \neq \emptyset$
   then $\{z \in \mathbb{R}^m : A^T z \geq 0 \text{ and } b^T z < 0\} = \emptyset$.

4. Consider the LPP:

$$\begin{aligned} & \max\,(x_1 + x_2) \\ s.t. \quad & \alpha\, x_1 + x_2 \leq \beta \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

   Determine $\alpha, \beta \in \mathbb{R}$, such that the LPP:

   (a) admits one optimal solution.

   (b) has a bounded feasible region.

   (c) has no feasible solution.

5. Consider the LPP:

$$\begin{aligned} &\max\,(x_1 + x_2)\\ s.t.\quad & 2x_1 + \ \ x_2 \leq 1\\ & x_1 + 5x_2 \leq 4\\ & 18x_1 + 9x_2 \geq 2\\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Solve the above LPP using the graphical method.

6. Consider the LPP:

$$\begin{aligned} &\max\,(2x_1 + x_2)\\ s.t.\quad & x_1 + 2x_2 \leq 14\\ & 2x_1 - \ \ x_2 \leq 10\\ & x_1 - \ \ x_2 \leq 3\\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Using the Simplex method in algebraic form, solve the LPP and plot the path of the extreme points, which increase the value of the objective function, around the boundary of the feasible region.

7. Solve the following LP problems using the Simplex method:

(a)

$$\begin{aligned} &\max\,(7x_1 + 5x_2 + 6x_3)\\ s.t.\quad & x_1 + \ \ x_2 - x_3 \leq 3\\ & x_1 + 2x_2 + x_3 \leq 8\\ & x_1 + \ \ x_2 \qquad\ \ \leq 5\\ & x_j \geq 0,\ j = 1, 2, 3. \end{aligned}$$

(b)

$$\begin{aligned} &\max\,(x_1 + x_2 + x_3 + x_4)\\ s.t.\quad & x_1 + x_2 + x_3 \qquad\ \ \leq 3\\ & x_2 + x_3 + x_4 \leq 4\\ & x_1 \qquad + x_3 + x_4 \leq 5\\ & x_1 + x_2 \qquad + x_4 \leq 6\\ & x_j \geq 0,\ j = 1, ..., 4. \end{aligned}$$

8. Apply the Two-phase method to solve the following LP problems with mixed constraints:

   (a)

$$\begin{aligned}
&\max\,(x_1 + x_2 + 3x_3)\\
&s.t.\quad x_1 + x_2 + x_3 \geq 100\\
&\qquad\qquad\ \ x_2 + x_3 \leq 80\\
&\qquad\quad x_1 \qquad + x_3 \leq 80\\
&\qquad\qquad\qquad x_j \geq 0,\ j = 1, 2, 3.
\end{aligned}$$

   (b)

$$\begin{aligned}
&\max\,(3x_1 - x_2 + 2x_3)\\
&s.t.\quad 3x_1 + 2x_2 - x_3 \leq 9\\
&\qquad\qquad\ 5x_2 - x_3 \leq 1\\
&\qquad 4x_1 - \ x_2 \qquad \geq 1\\
&\qquad\ x_1 + \ x_2 + x_3 \leq 3\\
&\qquad\qquad\qquad x_j \geq 0,\ j = 1, 2, 3.
\end{aligned}$$

9. A courier company has two aircrafts A_1 and A_2. We want to transfer 254 tonnes of products. In each flight, aircraft A_1 can transfer 22 tonnes with a cost of 12.000 Euros and fuel consumption 4.000 tonnes. Additionally, aircraft A_2 carries 12 tonnes with a cost of 10.000 Euros and fuel consumption 900 tonnes per flight. In total, 30.800 tonnes of fuel are consumed. How many aircrafts of each type should we choose so that the total cost is minimum.