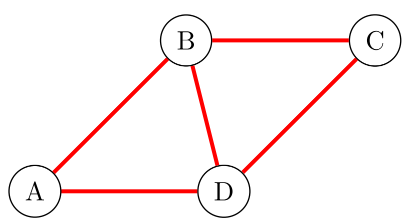
Exercise Sheet 2

1. Prove that the following propositions are equivalent:
   (a) A is total unimodular.
   (b) A^T is total unimodular.
   (c) (A, I) is total unimodular.

2. Show that the matrix
$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
is not total unimodular, but the solution of the linear system $Ax = b$, for every integral vector b, is integral.

3. Consider the following undirected graph $G = (V, E)$:



Construct the incidence matrix $A \in \{0, 1\}^{4\times 5}$ of G and check if it is total unimodular. If it is not, give an example of a graph $G_t = (V, E_t), E_t \subset E$, with $|E_t| = 3$, such that the incidence matrix A_t of G_t is total unimodular.

4. Consider the following two systems of inequalities:
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Show that both of them describe the same polyhedron and check if they are total dual integral.

5. Consider the family of inequalities $A_k x \leq b_k$, $k = 1, 2, ...$ where

$$\begin{pmatrix} -1 & 0 \\ 1 & 2k \\ 1 & -2k \end{pmatrix}, \quad \text{and} \quad b_k = \begin{pmatrix} 0 \\ 2k \\ 0 \end{pmatrix}.$$

   (a) Determine the polyhedron $P_k := \{x \in \mathbb{R}^2 : A_k x \leq b_k\}$ and find its integer hull P_I.

   (b) Show that $P_{k-1} \subseteq P'_k$ where P' denotes the 0-th Gomory–Chvátal truncation of P.

   (c) Consider higher order truncations $P_k^{(t)}$ for $t < k$, then show that

$$P_k^{(t)} \neq P_I.$$

6. Solve the following Integer Linear Programming Problems using the Gomory's cutting plane algorithm,

$$\begin{aligned} (a) \quad & \max\,(5x_1 + 2x_2) \\ & s.t. \quad 2x_1 + x_2 \leq 3 \\ & \quad -2x_1 + x_2 \leq 0 \\ & \quad x_1 \geq 0,\ x_2 \geq 0 \\ & \quad x_1,\ x_2 \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} (b) \quad & \max\,(x_1 + x_2) \\ & s.t. \quad x_1 + 2x_2 \leq 6 \\ & \quad 3x_1 + 2x_2 \leq 12 \\ & \quad x_1 \geq 0,\ x_2 \geq 0 \\ & \quad x_1,\ x_2 \in \mathbb{Z} \end{aligned}$$

7. Apply a greedy algorithm to find the minimum cost spanning tree and it's cost of the graph presented in Figure 1. Is it unique? Write the sequence in which the edges are added to the minimum spanning tree.

8. (Glo01) Consider the minimum cost spanning tree problem for the graph consisting of five vertices (Figure 2). The costs c_j of the edges e_j, $j = 1, ..., 7$ are indicated in parentheses. The edges are considered as $0-1$ variables, defined by

$$e_j = \begin{cases} 1, & \text{if edge } e_j \text{ is in the tree,} \\ 0, & \text{if edge } e_j \text{ is not in the tree.} \end{cases}$$

   We include also constraints that prohibit certain edges from appearing in the tree, namely,

$$\begin{aligned} e_1 + e_2 + e_6 &\leq 1 \\ e_1 &\leq e_3 \end{aligned}$$

   A greedy algorithm resulted to a first solution $\{e_1, e_4, e_5, e_6\}$ with total cost $c = 16$. Check if this solution is feasible. Apply a greedy algorithm with tabu search in order to obtain a local minimum. Consider three iterations, where each iteration consists of dropping an edge and adding another one. The tabu restriction is that the most recently added edge cannot be dropped.

9. The Best–in Greedy algorithm can result in different minimum spanning trees for the same input graph G, depending on the sorting of the edges with same weight. Modify the way to sort the edges of G, in the Greedy algorithm, such that for each minimum spanning tree T of G, the algorithm returns T. Write a pseudo–code.

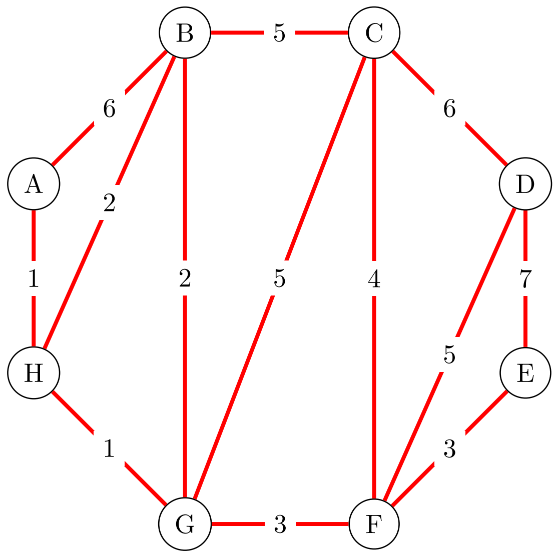


Figure 1: Undirected graph (Exercise 7).

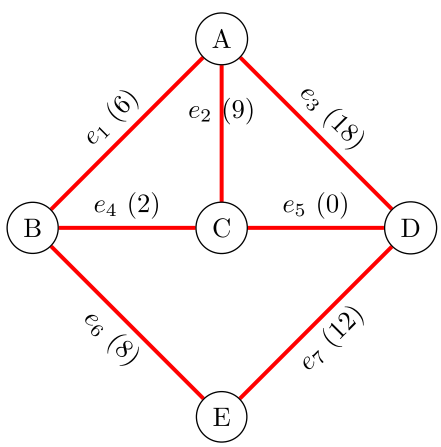


Figure 2: Undirected graph (Exercise 8).