Exercises Inverse Problems, SS 2010

Exercise Sheet 4

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Exercise 14. Let $\phi : [0, \infty) \to \mathbb{R}$ be such that $x \mapsto f_\phi(x) := \phi(|x|)$ is in $C_0^\infty(\mathbb{R}^2)$. Show that

$$(\mathbf{R}\, f_\phi)(n, r) = (\mathbf{A}\, \phi)(r) := 2 \int_r^\infty \frac{s\phi(s)}{\sqrt{s^2 - r^2}}\, ds\,.$$

The mapping $\phi \mapsto \mathbf{A}\, \phi$ is called Abel transform.

Exercise 15. Let ϕ be as in Exercise 14. Show that the inversion formula

$$\phi(r) = -\frac{1}{\pi} \int_r^\infty \frac{(\mathbf{A}\, \phi)'(s)}{\sqrt{s^2 - r^2}}\, ds\,.$$

holds. (Hint: Integrate the integrand in the definition of $\mathbf{A}\, \phi$ by parts. Then insert the candidate for the inverse and use the formula $\partial/\partial x \int_x^b g(x, u) du = -g(x, x) + \int_x^b \partial g/\partial x(x, u) du$.)

Exercise 16. The file `abel_test.m` contains an implementation of the Abel transform and its inverse. Explain the functions `abel(r,phi,N)` and `abel_inverse(r,psi,N)`. Test this file by varying `delta` and `dr`. One observes that even for `delta = 0` the vectors `phirek` and `phi` are different. What does this tell us?

Exercise 17. The implementations in Example 16 use the approximation

$$\int_r^\infty \frac{\phi(s)}{\sqrt{s^2 - r^2}} =: I(r) \simeq I_h(r) := h \sum_{i=1}^\infty \frac{\phi(r + ih)}{\sqrt{(r + ih)^2 - r^2}}\,.$$

Show that $\lim_{h \to 0} I_h(r) = I(r)$. Try to estimate the behavior of $e(r, h) = |I(r) - I_h(r)|$ as $h \to 0$.

Exercise 18. Improve `abel(r,phi,N)` and `abel_inverse(r,psi,N)` based on the approximation

$$I(r) \simeq h \sum_{i=1}^\infty \frac{\phi(r + ih)}{r + ih} \int_{r+(i-1)h}^{r+ih} \frac{s\, ds}{\sqrt{s^2 - r^2}}\,.$$

Exercise 19. Let X be a Banach space with norm $\|\cdot\|$, let $Y \subset X$ be a closed subspace of X, and let $\epsilon \in (0, 1)$. Show, that there exists $x \in X$ such that $\|x\| = 1$ and $\|y - x\| \geq \epsilon$ for all $y \in Y$.

Exercise 20. Let X be a Banach space. Show that the identity operator $\mathbf{I} : X \to X$ is compact, if and only if X is finite dimensional.