Exercises Inverse Problems, SS 2010

Exercise Sheet 5 (Sequence Spaces)

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Exercise 21. For $1 \leq p < \infty$, let

$$\ell^p(\mathbb{N}) := \left\{ x := (x_1, x_2, \dots) \subset \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^p < \infty \right\}$$

denote the set of all real valued sequences x with $\sum_{j=1}^{\infty} |x_j|^p < \infty$. Moreover denote by

$$\mathbb{R}^{(\mathbb{N})} := \left\{ x = (x_1, x_2, \dots) \subset \mathbb{R} : \{ j \in \mathbb{N} : x_j \neq 0 \} \text{ is finite } \right\}$$

the set of all terminateing sequences.

(a) Show that $\ell^p(\mathbb{N})$ (with pointwise addition and scalar multiplication) is a Banach space with norm $\|x\|_p := \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}$.

<u>Hint:</u> To show that $\|\cdot\|_p$ is a norm, use that $\left(\sum_{j=1}^{N} |x_j|^p \right)^{1/p}$ is a norm on $\mathbb{R}^N$. To show that $\ell^p(\mathbb{N})$ is complete first show that any Cauchy sequence (x^k) in $\ell^p(\mathbb{N})$ converges pointwise to a sequence x and then verify $\lim_k \left\| x^k - x \right\|_p = 0$.

(b) Show that $\mathbb{R}^{(\mathbb{N})}$ is dense in $\left(\ell^p(\mathbb{N}), \|\cdot\|_p \right)$.

Exercise 22. Let

$$\ell^\infty(\mathbb{N}) := \{ x := (x_1, x_2, \dots) \subset \mathbb{R} : \sup \{ |x_j| : j \in \mathbb{N} \} < \infty \}$$

denote the set of all bounded real valued sequences. Show that $\ell^\infty(\mathbb{N})$ is a Banach space with norm $\|x\|_\infty := \sup \{ |x_j| : j \in \mathbb{N} \}$. Is $\mathbb{R}^{(\mathbb{N})}$ is dense in $\left(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty \right)$?

Exercise 23. Denote

$$c(\mathbb{N}) := \left\{ x := (x_1, x_2, \dots) \subset \mathbb{R} : \lim_{j \to \infty} x_j < \infty \right\},$$

$$c_0(\mathbb{N}) := \left\{ x := (x_1, x_2, \dots) \subset \mathbb{R} : \lim_{j \to \infty} x_j = 0 \right\}.$$

(a) Show that $c(\mathbb{N})$ and $c_0(\mathbb{N})$ are a closed subspaces of $\left(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty \right)$

(b) Show that $c_0(\mathbb{N})$ is a closed subspaces of $\left(c(\mathbb{N}), \|\cdot\|_\infty \right)$. What is the co-dimension of $c_0(\mathbb{N})$ in $c(\mathbb{N})$? (The co-dimension of Y in X is the dimension of the quotient X/Y.)

(c) What is the closure of $\mathbb{R}^{(\mathbb{N})}$ in $\left(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty \right)$?

Exercise 24. For $x = (x_1, x_2, \dots) \in \ell^1(\mathbb{N})$ define

$$\Phi_x : c_0(\mathbb{N}) \to \mathbb{R} : y = (y_1, y_2, \dots) \mapsto \langle x, y \rangle := \sum_j x_j y_j \,.$$

show that $\Phi : \ell^1(\mathbb{N}) \to c_0(\mathbb{N})^* : x \mapsto \Phi_x$ is well defined and an isometric isomorphism. Explain why this amounts to write $c_0(\mathbb{N})^* = \ell^1(\mathbb{N})$.
<u>Hint</u>: (1) First show that Φ is an isometry. To that end fofirst verify that $\|\Phi_x\| \le \|x\|_1$ and then find, for positive ϵ, an element $y \in \mathbb{R}^{(\mathbb{N})}$ with $\|\Phi_x(y)\| \ge \|x\|_1 - \epsilon$. Such an element is constructed by setting $y_j = \mathrm{sign}(x_j)$ for $j \le n(\epsilon)$ for some proper chosen index $n(\epsilon)$. (2) To show that Φ is surjective show that for $\psi \in c_0(\mathbb{N})^*$ and $y = (\psi(e_j))_{j \ge 1}$ one has $\Phi_x(y) = \psi$. (Here e_j is the j-th standart basis vector, $(e_j)_k = \delta_{j,k}$).

Exercise 25. Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. Show that (in the sense of Exercise 24) one has $\ell^p(\mathbb{N})^* = \ell^q(\mathbb{N})$ and $\ell^1(\mathbb{N})^* = \ell^\infty(\mathbb{N})$

Exercise 26. Let $(\sigma_j)_{j \in \mathbb{N}}$ be a bounded sequence of real numbers. Show that $\Sigma : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by $(\Sigma x)_j := \sigma_j x_j$ is well defined, linear and bounded. Find the adjoint of Σ. What is the inverse of Σ? What happens if (σ_j) is unbounded? What if Σ is considered as a mapping on $\ell^p(\mathbb{N})$?