Exercise Sheet 1

1. We define the Sobolev space $H^1([a,b])$ as the set of all functions $f:[a,b]\to\mathbb{R}$ of the form
$$f(x)=\sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{\frac{2\pi}{b-a}\mathrm{i}kx}$$
with $(f_k)_{k\in\mathbb{Z}}\in\ell^2(\mathbb{C})$ and $(\mathrm{i}kf_k)_{k\in\mathbb{Z}}\in\ell^2(\mathbb{C})$.
Let $\Delta=(x_i)_{i=0}^l$ be a mesh on $[a,b]$.

   (a) Show that a step function
$$f:[a,b]\to\mathbb{R},\quad f(x)=\sum_{i=1}^{l} s_i\chi_{[x_{i-1},x_i)}(x)$$
is in $H^1([a,b])$ if and only if it is a constant function. So, the spline space $S_{0,\Delta}$ is not a subspace of $H^1([a,b])$.

   (b) Let Λ_i, $i\in\{0,\dots,l\}$, be the continuous function which is linear on every interval $[x_{k-1},x_k]$, $k=1,\dots,l$, and has the values $\Lambda_i(x_j)=\delta_{ij}$, $j=0,\dots,l$. Show that
$$\Lambda_i\in H^1([a,b])$$
and conclude that
$$S_{1,\Delta}\subset H^1([a,b]).$$

2. We consider the function
$$f:[-1,1]\to\mathbb{R},\quad f(x)=x^2.$$
Find the interpolating and the best approximating linear spline $s\in S_{1,\Delta}$ of f on the mesh $\Delta=(-1,0,1)$.

3. Let $\Delta=(x_i)_{i=0}^l$ be a mesh on $[a,b]$, and let $f\in H^1([a,b])$. We define the step function $s\in S_{0,\Delta}$ by
$$s(x)=\sum_{i=1}^{l} s_i\chi_{[x_{i-1},x_i)}(x),\quad s_i=f\left(\frac{x_{i-1}+x_i}{2}\right),\quad i=1,\dots,l.$$
Prove the error estimate
$$\|f-s\|_{L^2([a,b])}\le\|f'\|_{L^2([a,b])}\max_{i\in\{1,\dots,l\}}(x_i-x_{i-1}).$$

4. Show that for every non-constant function $f \in C^1([a,b])$, there exists a constant $c > 0$ such that we have for every mesh $\Delta = (x_i)_{i=0}^l$ on $[a,b]$ the estimate

$$\|f - s\|_{L^2([a,b])} \geq c \min_{i \in \{1,\dots,l\}} (x_i - x_{i-1}) \quad \text{for all} \quad s \in S_{0,\Delta}.$$

5. We consider the interval $[-2,2]$ with the mesh $\Delta = (-2,-1,-\frac{1}{2},0,\frac{1}{2},1,2)$. We define the step function $\beta_0 = \chi_{[-\frac{1}{2},\frac{1}{2})} \in S_{0,\Delta}$. Show that the convolution $\beta_0 * \beta_0$ is a linear spline and that $\beta_0 * \beta_0 * \beta_0 * \beta_0$ is a cubic spline on this mesh. Sketch the shape of these splines.

6. Write a program which calculates for a given mesh $\Delta = (x_i)_{i=0}^l$ and given values $(y_i)_{i=0}^l$ the moments $(\gamma_j)_{j=1}^{l-1}$ of the natural cubic spline $s \in S_{3,\Delta}$ with $s(x_i) = y_i$, $i = 0, \dots, l$.

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