Exercise Sheet 7

1. We consider the Sturm-Liouville problem

$$\begin{aligned} -(au')'(x) &= f(x), \quad x \in (0,1), \\ u(0) = u(1) &= 0, \end{aligned}$$

for the function $u \in C^2([0,1])$ for some positive function $a \in C^3([0,1])$ and some function $f \in C^2([0,1])$.

What is the order of consistency of the finite difference method

$$\begin{aligned} -D_{\frac{h}{2}}[aD_{\frac{h}{2}}[u]](x_i) &= f(x_i), \quad i = 1, \dots, n-1, \\ u(x_0) = u(x_n) &= 0 \end{aligned}$$

on the uniform mesh $(x_i)_{i=0}^n$ on $[0,1]$ with step size $h = \frac{1}{n} \in (0, \frac{1}{2})$? (Here,

$$D_{\frac{h}{2}}[v](x) = \frac{v(x+\frac{h}{2}) - v(x-\frac{h}{2})}{h}, \quad x \in [\tfrac{h}{2}, 1-\tfrac{h}{2}],$$

denotes the central difference quotient of a function $v : [0,1] \to \mathbb{R}$.)

2. We consider the finite difference method

$$\begin{aligned} -\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + b(x_i)\frac{u_{i+1} - u_i}{h} + c(x_i)u_i &= f(x_i), \quad i = 1, \dots, n-1, \\ u_0 = u_n &= 0 \end{aligned}$$

for the boundary value problem

$$\begin{aligned} -u''(x) + b(x)u'(x) + c(x)u(x) &= f(x), \quad x \in (0,1), \\ u(0) = u(1) &= 0 \end{aligned}$$

for given functions $b, c, f \in C^2([0,1])$ with $c(x) \geq 0$ for all $x \in [0,1]$ on the uniform mesh $(x_i)_{i=0}^n$ on $[0,1]$ with step size $h = \frac{1}{n} \in (0, \frac{1}{2})$.

(a) Prove that this finite difference method is for every h stable if $b(x) \leq 0$ for all $x \in [0,1]$.

(b) Show that this is no longer necessarily true if the function b can also take positive values.

3. Let $A \in \mathbb{R}^{n\times n}$ be a real matrix with $A_{ij} \le 0$ for all $i \ne j$. Prove the equivalence of the following statements:

   (i) A is an M-matrix,

   (ii) there exists a vector $v \in \mathbb{R}^n$ with $v_i > 0$ and $(Av)_i > 0$ for all i such that
   $$\|A^{-1}\|_\infty \le \frac{\|v\|_\infty}{\min_{1\le i\le n}(Av)_i},$$

   (iii) setting $A = D - N$, where D shall be the diagonal matrix with the entries $D_{ii} = A_{ii}$ for all i, we have that $D_{ii} > 0$ for all i and that the spectral radius
   $$\rho(D^{-1}N) = \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } D^{-1}N\}$$
   of the matrix $D^{-1}N$ is strictly less than one.

4. Consider the problem
   $$\begin{aligned} u''(x) &= \gamma u(x), \quad x \in (0,1) \\ u(0) &= 1, \\ u(1) &= 0 \end{aligned}$$
   for some constant $\gamma > 0$. We define the function $F : \mathbb{R} \to \mathbb{R}$ by $F(\alpha) = v_\alpha(1)$ where $v_\alpha \in C^2([0,1])$ is the solution of the inital value problem
   $$\begin{aligned} v_\alpha''(x) &= \gamma v_\alpha(x), \quad x \in (0,1), \\ v_\alpha'(0) &= \alpha, \\ v_\alpha(0) &= 1. \end{aligned}$$
   Determine for given $\varepsilon > 0$ the maximal error $\delta > 0$ so that
   $$|F(\alpha)| < \varepsilon \quad \text{for all} \quad \alpha \in (F^{-1}(0) - \delta, F^{-1}(0) + \delta).$$
   What does this mean for the shooting method for this problem if γ is large?

5. Let us consider the non-linear boundary value problem
   $$\begin{aligned} u''(x) + u(x)u'(x) &= -1, \quad x \in (0,1), \\ u(0) = u(1) &= 0 \end{aligned}$$
   for the function $u \in C^2([0,1])$. We define the map $F : \mathbb{R} \to \mathbb{R}$ by $F(\alpha) = v_\alpha(1)$ where v_α is defined as the solution of the initial value problem
   $$\begin{aligned} v_\alpha''(x) + v_\alpha(x)v_\alpha'(x) &= -1, \quad x \in (0,1), \\ v_\alpha'(0) &= \alpha, \\ v_\alpha(0) &= 0. \end{aligned}$$

(a) Show that $F'(\alpha) = w_\alpha(1)$ where w_α is a solution of the initial value problem

$$\begin{aligned} w''_\alpha(x) + v_\alpha(x)w'(x) + v'_\alpha(x)w_\alpha(x) &= 0, \quad x \in (0,1), \\ w'_\alpha(0) &= 1, \\ w_\alpha(0) &= 0. \end{aligned}$$

(b) Rewrite the second order initial value problems for v_α and w_α as a system of first order initial value problems.

(c) Write a program that solves the boundary value problem by using the Newton method

$$\alpha_{k+1} = \alpha_k - \frac{F(\alpha_k)}{F'(\alpha_k)}, \quad k \in \mathbb{N}_0,$$

for a given starting point $\alpha_0 \in \mathbb{R}$ to find a zero of the function F. (Pick your favourite Runge-Kutta method to solve the system of first order initial value problems obtained in part (b).)

—