Exercise Sheet 4

1. Let $f(x) = x^5 + x^4$ and the grid $\{-2, -1, 0, 1, 2\}$ on the interval $[-2, 2]$. Determine the natural cubic spline which interpolates the function f at the grid points.

2. Consider the integral
$$\int_1^5 \frac{1}{x}\,dx.$$
Approximate the value of the integral using the Trapezoidal rule and the composite Simpson rule for $n = 4$ sub-intervals. Which rule provides a better approximation to the exact value of the integral?

3. Consider the quadrature rule
$$Q(f) = w_0 f(-1) + w_1 f(0) + w_2 f(1)$$
that estimates the integral
$$I(f) \equiv \int_{-1}^{1} f(x)\,dx.$$
   (a) Determine the weights w_0, w_1 and w_2 such that $Q(f)$ is exact for polynomials of degree 3.

   (b) Peano's theorem tell us that for $f \in C^4[a,b]$, there exist $\eta \in (-1,1)$ such that
   $$I(f) - Q(f) = \kappa\, f^{(4)}(\eta),$$
   where $f^{(4)}$ denotes the fourth derivative of f. Compute the Peano's constant κ considering the special choice $f(x) = x^4$.

4. Consider the initial-value problem
$$y'(t) = 1 + (t - y(t))^2, \quad t \in [2,3], \quad y(2) = 1,$$
with exact solution
$$y(t) = t + \frac{1}{1-t}.$$
Apply the Euler method to approximate y setting as grid points $t_i := 2 + i/2$, $i = 0, 1, 2$. In each step, compute also the error $\epsilon_i := |y_i - y(t_i)|$.

5. Let $n \in \mathbb{N}$, $h = (b-a)/n$ and $x_i := a + i\,h$, $i = 0, ..., n$. Consider the quadrature formula,

$$Q_{n+1}(f) := h\left[\frac{1}{2}f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}f(x_n)\right] - \frac{h^2}{12}\left[f'(x_n) - f'(x_0)\right],$$

for $f \in C^1[a,b]$. Implement the above formula in a MATLAB-Program and find the minimum value of n such that

$$\int_a^b f(x)dx - Q_{n+1}(f) \leq 10^{-5},$$

is satisfied for $f(x) = e^{2x}$, $a = 0$ and $b = 1$.

6. Create a MATLAB-Program that implements the composite Simpson rule for approximating the integral of $f(x) = e^{-x^2}$ at the interval $[0,1]$. How many nodal points are required for an accuracy of 6 decimal places? Compare this algorithm with the trapezoidal rule (MATLAB-Function `trapz`), i.e. how many nodal points are needed (approximately) to obtain the same accuracy using the trapezoidal rule.