Exercise Sheet 6

1. Consider an appropriate iterative method fro solving the following system for $x, y \in \mathbb{C}$,

$$(2-3i)x + (1+2i)y = 3 - i$$
$$(1+3i)x + (2+2i)y = 2$$

2. Consider the system

$$Ax = b,$$

where the pentadiagonal matrix $A = \text{pentadiag}(-1, -1, 10, -1, -1) \in \mathbb{R}^{10\times 10}$ is decomposed to $A = M + N + D$, where $D = \text{diag}(8, ..., 8) \in \mathbb{R}^{10\times 10}$, $M = \text{pentadiag}(-1, -1, 1, 0, 0) \in \mathbb{R}^{10\times 10}$ and $N = M^T$. To solve the above linear system we consider the following two iterative methods:

   (a) $Dx^{(n+1)} = -(M + N)x^{(n)} + b$

   (b) $(M + N)x^{(n+1)} = -Dx^{(n)} + b$

   Analyse the convergence of both methods.

3. Let

$$f(x) = x^3 - 3x^2 + x - 1.$$

Show that there exist an interval $I \subset \mathbb{R}$ such that the Newton method for the function f and an arbitrary initial value $x_0 \in I$ diverges.

4. Check if the iterative methods, Jacobi and Gauss-Seidel, converge to the solution of the linear equation $Ax = b$ for $b \in \mathbb{R}^{4\times 4}$ and

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 5/2 & 0 & -1 \\ -1 & 0 & 5/2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

5. Solve the following system using the Conjugate Gradient method,

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

6. Consider the matrix A of the particular form

$$A = \left(\begin{array}{cccc|cccc} a_1 & b_1 & & & & & & \\ b_1 & \ddots & \ddots & & & & & \\ & \ddots & \ddots & b_{m-1} & & & & \\ & & b_{m-1} & a_m & b_m & & & \\ \hline & & & b_m & a_{m+1} & b_{m+1} & & \\ & & & & b_{m+1} & \ddots & \ddots & \\ & & & & & \ddots & \ddots & b_{n-1} \\ & & & & & & b_{n-1} & a_n \end{array}\right)$$

and the two submatrices

$$A_1 = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{m-1} \\ & & b_{m-1} & c_{m-1} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} c_m & b_{m+1} & & \\ b_{m+1} & a_{m+2} & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{pmatrix}$$

where $c_{m-1} = a_m - b_m$ and $c_m = a_{m+1} - b_m$. We consider the vectors $w = e_m + e_{m+1} \in \mathbb{R}^n$, where e_j are the unit vectors in $\mathbb{R}^n$ and we rewrite A in the form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + b_m w w^T$$

Given the eigen-decompositions $A_j = V_j \Lambda_j V_j^T$, for $j = 1, 2$, (orthonormal eigenvectors) find the matrices V, Λ and the vector z such that

$$V^T A V = \Lambda + b_m z z^T$$

A method to find the eigenvalues of A is by solving the equation

$$f(\lambda) = 1 + b_m \sum_{j=1}^{n} \frac{z_j^2}{\Lambda_{jj} - \lambda} = 0. \qquad (1)$$

Consider the above analysis to find the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 3 & & \\ 3 & -6 & 1 & \\ & 1 & 3 & 1 \\ & & 1 & 2 \end{pmatrix}.$$

Solve equation (1) with MATLAB and plot the graph of f.