Exercise Sheet 2

1. Compute the LR-decomposition of the matrix

$$A = \begin{bmatrix} 6 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}.$$

2. Consider the linear systems

$$\begin{bmatrix} 0 & 4 & -1 \\ 3 & -1 & -1 \\ 1 & -5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \\ 11 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

Solve the above systems using Gauss-elimination with partial pivoting.

3. Compute the Cholesky-decomposition of the matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

4. Let

$$A = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

Find the inverse of A using the Cholesky-decomposition.

5. Let $v = [2, -2, 1]^T$. Compute the Householder transformation $H \in \mathbb{R}^{3\times 3}$ and verify that

a) $H^2 v = v$

b) $\|Hv\|_2 = \|v\|_2$

c) $(Hw)^T Hv = 0, \quad \text{where} \quad w = [0, 1, 2]^T.$

6. Consider the linear system $Ax = b$ with

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 & a_{1n} \\ 0 & a_{22} & \ddots & \vdots & a_{2n} \\ \vdots & \ddots & a_{33} & 0 & a_{3n} \\ 0 & \cdots & 0 & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n.$$

Write a MATLAB-Function, that for given matrix A (of the above form) and vector b solves the linear system using the substitution method. This method works as follows: We solve the equations (except one) for one of the variables (you choose which one) and then plugging this back into the last equation, "substituting" for the chosen variables and solving for the last one. The remaining variables are computed from the previous equations.

7. We define the **golden ratio** ϕ as the positive solution of the quadratic equation $\phi^2 - \phi - 1 = 0$. Using one of the following iteration methods

$$a) \quad \phi_{n+1} = 1 + \frac{1}{\phi_n}$$

or

$$b) \quad \phi_{n+1} = \frac{\phi_n^2 + 1}{2\phi_n - 1}$$

we can approximate the golden ratio as $\phi = \lim_{n\to\infty} \phi_n$. We set as initial value $\phi_0 = 1$. The iterations are terminated if the stopping criterion

$$\|\phi^* - \phi_n\|_2 \leq 10^{-4}, \quad \text{where} \quad \phi^* \quad \text{is the exact solution,}$$

is satisfied. Create a MATLAB-Program that implements the two iteration schemes for approximating ϕ. Which iteration method converges faster?