Exercise Sheet 4

1. Consider the initial-value problem

$$y'(t) = 1 + (t - y(t))^2, \quad t \in [2,3], \quad y(2) = 1,$$

with exact solution

$$y(t) = t + \frac{1}{1-t}.$$

Apply the Euler method to approximate y setting as grid points $t_i := 2 + i/2$, $i = 0, 1, 2$. In each step, compute also the error $\epsilon_i := |y_i - y(t_i)|$.

2. Consider the quadrature rule

$$Q(f) = w_0 f(-1) + w_1 f(0) + w_2 f(1)$$

that estimates the integral

$$I(f) \equiv \int_{-1}^{1} f(x)\, dx.$$

(a) Determine the weights w_0, w_1 and w_2 such that $Q(f)$ is exact for polynomials of degree 3.

(b) Peano's theorem tells us that for $f \in C^4[a,b]$, there exist $\eta \in (-1,1)$ such that

$$I(f) - Q(f) = \kappa\, f^{(4)}(\eta),$$

where $f^{(4)}$ denotes the fourth derivative of f. Compute the Peano's constant κ considering the special choice $f(x) = x^4$.

3. Consider the following Runge-Kutta arrays

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

which define two second-order Runge-Kutta methods for approximating the solution of the initial value problem

$$y'(t) = -y(t), \quad t > 0, \quad y(0) = 1$$

For a given $h > 0$, find for both arrays the coefficients $C(h)$, such that the corresponding method takes the form

$$y_{i+1} = C(h)\, y_i$$

4. Consider the Runge-Kutta method with tableau

$$\begin{array}{c|cc} \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\ \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Show that this method is A-stable.

5. Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad t \in [t_0, b],$$

and it's perturbation

$$y'_\epsilon(t) = f(t, y_\epsilon(t)), \quad y_\epsilon(t_0) = y_0 + \epsilon,\ \epsilon > 0, \quad t \in [t_0, b],$$

An initial value problem is considered to be well-conditioned if

$$\|y_\epsilon - y\|_\infty = \max_{0\le t\le b} |y_\epsilon(t) - y(t)| \le c\,\epsilon,$$

for some $c > 0$ independent of ϵ. Consider the problems

(a)

$$y'(t) = \lambda(y(t) - 1), \quad \lambda \in \mathbb{R}, \quad t \in [0, b],$$

with general solution

$$y(t) = 1 + c_a\, e^{\lambda t}, \quad c_a \in \mathbb{R}.$$

(b)

$$y'(t) = -y^2(t), \quad t \in [0, b],$$

with general solution

$$y(t) = \frac{1}{t - c_b}, \quad c_b \in \mathbb{R}.$$

Set as initial condition $y(0) = 1$ in both of the problems and characterize them with respect to stability.

6. Let $n \in \mathbb{N}$, $h = (b-a)/n$ and $x_i := a + i\,h$, $i = 0, ..., n$. Consider the quadrature formula,

$$Q_{n+1}(f) := h\left[\frac{1}{2}f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}f(x_n)\right] - \frac{h^2}{12}\left[f'(x_n) - f'(x_0)\right],$$

for $f \in C^1[a, b]$. Implement the above formula in a MATLAB-Program and find the minimum value of n such that

$$\int_a^b f(x)dx - Q_{n+1}(f) \le 10^{-5},$$

is satisfied for $f(x) = e^{2x}$, $a = 0$ and $b = 1$.

7. Consider the trapezoidal method

$$y_{i+1} = y_i + \frac{t_{i+1} - t_i}{2} \left[f(t_i, y_i) + f(t_{i+1}, y_{i+1}) \right],$$

to approximate the solution of the initial value problem

$$y'(t) = f(t, y(t)), \quad t \in [a, b], \quad y(t_0) = y_0,$$

in $n + 1$ equidistant points in $[a, b]$.

Implement in MATLAB the Euler method and the trapezoidal method to approximate the exact solution $y(t) = e^{t - t^2/2}$ of the initial value problem

$$y'(t) = (1 - t) y(t), \quad y(0) = 1, \quad t \in [0, 2],$$

for $h := t_{i+1} - t_i = 0.5$, 0.2 and 0.1.