Übungen zu Numerische Methoden I

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Exercise Sheet 5

1. Consider the piecewise function

$$f(x) = \begin{cases} 0, & 0 \le x \le 1 \\ (x-1)^4, & 1 < x \le 2 \end{cases}$$

and the piecewise polynomial

$$p(x) = \begin{cases} 0, & 0 \le x \le 1 \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3, & 1 < x \le 2 \end{cases}$$

We approximate the function f in $[0,2]$ with the polynomial p. Compute the coefficients a, b, c, d if $p \in C^1[0,2]$ and

$$p(0) = f(0),\ p'(0) = f'(0),\ p(1) = f(1),\ p(2) = f(2),\ p'(2) = f'(2).$$

2. Consider the quadrature rule

$$Q(f) = w_0 f(0) + w_1 f(\pi) + w_2 f(2\pi)$$

that estimates the integral

$$I(f) \equiv \int_0^{2\pi} f(x) \sin(x)\, dx.$$

Determine the weights w_0, w_1 and w_2 such that $Q(f)$ is exact for polynomials of degree 2.

3. Let the linear system of ODEs

$$\begin{aligned} y_1'(t) &= -100\, y_1(t), \quad y_1(0) = 1 \\ y_2'(t) &= -2\, y_2(t) + y_1(t), \quad y_2(0) = 1 \end{aligned}$$

Characterize the above system with respect to stiffness.

4. Consider the differential equation

$$y'(t) = Ay(t), \quad A \in \mathbb{R}$$

with solution $y(t) = e^{At}$. Show that the Euler method, for small h, converges to the exact solution.

Hint: You should show that $y_{n+1} \simeq e^{At_{n+1}}$ using an approximation for the exponential and writing the $(n+1)$th step and h with respect to the initial value $y_0 = e^{At_0}$ and the initial point t_0, respectively.

5. Create a MATLAB-Program that implements the composite Simpson rule for approximating the integral of $f(x) = e^{-x^2}$ at the interval $[0, 1]$. How many nodal points are required for an accuracy of 6 decimal places? Compare this algorithm with the trapezoidal rule (MATLAB-Function `trapz`), i.e. how many nodal points are needed (approximately) to obtain the same accuracy using the trapezoidal rule.

6. The fourth-order Runge-Kutta method is given by

$$\begin{aligned}
y_0 &= y(a), \\
k_0 &= hf(t_i, y_i), \\
k_1 &= hf(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_0), \\
k_2 &= hf(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_1), \\
k_3 &= hf(t_{i+1}, y_i + k_2), \\
y_{i+1} &= y_i + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3), \quad i = 0, ..., n-1.
\end{aligned}$$

Implement in MATLAB the above method to approximate the solution of the initial value problem

$$y'(t) = -\frac{y(t)}{1+t}, \quad t \in [0, 1], \quad y(0) = 1, \quad \text{for} \quad h = 0.005.$$

7. Solve the equation

$$y'(t) = -50(y(t) - \cos t), \quad t > 0$$

using the explicit and the implicit Euler method in MATLAB for $t \in [0, 1.5]$. Compare the results for different step sizes $h = 0.05, 0.1$ and 0.5. Plot the results, for the initial value $y_0 = 0.15$, compared to the smooth solution $y \approx \cos t$.