PDEs in Image Processing, Tutorials

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Direct Methods

Let X be a topological space and $\mathcal{R}\colon X \to \mathbb{R} \cup \{+\infty\}$ some functional. Recall the following definitions:

- The mapping $\mathcal{R}$ is *lower semi-continuous*, if for every $t \in \mathbb{R}$ the set
$$\operatorname{level}_t(\mathcal{R}) := \bigl\{x \in X : \mathcal{R} \le t\bigr\}$$
is closed.
- The mapping $\mathcal{R}$ is *coercive*, if for every $t \in \mathbb{R}$ the set $\operatorname{level}_t(\mathcal{R})$ is pre-compact.

1. Let $\mathcal{R}_i\colon X \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, be a family of functionals on the topological space X. Define moreover $\mathcal{R} := \sup_i \mathcal{R}_i$, that is, $\mathcal{R}(x) = \sup_i \mathcal{R}_i(x)$ for every $x \in X$. Show that the following assertions hold:
   a) If every functional $\mathcal{R}_i$ is lower semi-continuous, then so is $\mathcal{R}$.
   b) If any functional $\mathcal{R}_i$ is coercive, then so is $\mathcal{R}$.

2. Let $\mathcal{R}$, $\mathcal{S}\colon X \to \mathbb{R} \cup \{+\infty\}$ be functionals on the topological space X. Show that the following assertions hold:
   a) If $\mathcal{R}$ and $\mathcal{S}$ are lower semi-continuous, then so is $\mathcal{R} + \mathcal{S}$.
   b) If $\mathcal{R}$ is coercive and $\mathcal{S}$ is bounded below (that is, $\inf_{x\in X} \mathcal{S}(x) > -\infty$), then $\mathcal{R} + \mathcal{S}$ is coercive.

3. Let X be a Hilbert space and $\mathcal{R}\colon X \to \mathbb{R} \cup \{+\infty\}$ any functional. Show that $\mathcal{R}$ is weakly coercive (that is, coercive with respect to the weak topology on X), if and only if
$$\sup_{t>0} \inf\bigl\{\mathcal{R}(u) : \|u\| \ge t\bigr\} = +\infty\ .$$

4. Show that the norm on a Hilbert space X is weakly lower semi-continuous and weakly coercive.

5. Let X be a topological space and $\mathcal{R}\colon X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous and coercive functional. Show that $\mathcal{R}$ attains its minimum on X.

Tikhonov Regularization

For the following exercises, we consider the following setting:

- The sets X and Y are Hilbert spaces, and $\mathcal{F}\colon \mathcal{D}(\mathcal{F}) \subset X \to Y$ is an operator mapping a subset of X to Y.
- The operator $\mathcal{F}$ is weakly closed in the sense that

  $$\operatorname{graph}(\mathcal{F}) := \big\{(u,y) \in X \times Y : u \in \mathcal{D}(\mathcal{F}),\ \mathcal{F}(u) = y\big\}$$

  is a weakly closed subset of $X \times Y$.
- Let $u_0 \in X$ be fixed. For every $\alpha > 0$ and $y \in Y$ the Tikhonov functional $\mathcal{T}_{\alpha,y}\colon X \to [0,+\infty]$ is defined by

  $$\mathcal{T}_{\alpha,y}(u) := \begin{cases} \|\mathcal{F}(u) - y\|^2 + \alpha\|u - u_0\|^2 & \text{if } u \in \mathcal{D}(\mathcal{F})\,, \\ +\infty & \text{else.} \end{cases}$$

6. For every $\alpha > 0$ and every $y \in Y$ the Tikhonov functional $\mathcal{T}_{\alpha,y}$ has a minimizer.
7. Let $\alpha > 0$ be fixed and let $\{y^{(k)}\}_{k\in\mathbb{N}} \subset Y$ be some sequences converging to $y \in Y$. Let moreover

   $$u^{(k)} \in \arg\min\big\{\mathcal{T}_{\alpha,y^{(k)}}(u) : u \in X\big\}\ .$$

   Then the sequence $\{u^{(k)}\} \subset X$ has a convergent subsequence. Moreover, the limit of every converging subsequence is a minimizer of the Tikhonov functional $\mathcal{T}_{\alpha,y}$.
8. Let $y \in Y$ be fixed. For every $\delta > 0$ let $y^\delta \in Y$ be such that $\|y - y^\delta\| \le \delta$. Let moreover $\alpha\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be such that $\lim_{\delta\to0}\alpha(\delta) = 0$ and $\lim_{\delta\to0}\delta^2/\alpha(\delta) = 0$. Let moreover $\{\delta^{(k)}\}_{k\in\mathbb{N}}$ be any sequence converging to 0 and let

   $$u^{(k)} \in \arg\min\big\{\mathcal{T}_{\alpha(\delta^{(k)}),y^{\delta^{(k)}}}(u) : u \in X\big\}\ .$$

   Then the sequence $\{u^{(k)}\}_{k\in\mathbb{N}}$ has a convergent subsequence. Moreover, the limit of every convergent subsequence is a u_0-minimum norm solution of the equation $\mathcal{F}(u) = y$.
9. Show that the sequence $\{u^{(k)}\}_{k\in\mathbb{N}}$ in Exercise 7 itself converges to a minimizer of the Tikhonov functional $\mathcal{T}_{\alpha,y}$ if the latter is unique. Similarly, if $u^\dagger$ is the unique u_0-minimum norm solution of the equation $\mathcal{F}(u) = y$, show that the sequence $\{u^{(k)}\}_{k\in\mathbb{N}}$ in Exercise 8 converges to $u^\dagger$.

Image Denoising: Linear Filters

One of the basic problems in image processing is that of denoising. Here one is given a noisy image $f \in L^2(\Omega)$ with $\Omega \subset \mathbb{R}^2$ sufficiently regular, which, one assumes, consists of an underlying true image u to which noise n has been added. That is, one has the equality

$$f = u + n \, .$$

In order to reconstruct u from f, one possibility is to minimize a Tikhonov like functional of the form

$$\mathcal{T}_{\alpha,f}(u) := \|u - f\|^2_{L^2(\Omega)} + \alpha \mathcal{R}(u) \,, \tag{1}$$

where the *regularization term* $\mathcal{R}$ should be chosen in such a way that "true" images are hardly penalized, while noise is penalized rather heavily.

10. Define the set $D \subset L^2(\Omega)$ by

$$u \in D \quad :\Longleftrightarrow \quad \sup\Bigl\{ \langle u, \operatorname{div} \vec{\varphi} \rangle_{L^2(\Omega)} : \vec{\varphi} \in C_c^1(\Omega;\mathbb{R}^2),\ \|\vec{\varphi}\|_{L^2(\Omega;\mathbb{R}^2)} \le 1 \Bigr\} < +\infty \, .$$

Show that there exists a unique (unbounded) linear mapping $\nabla \colon D \subset L^2(\Omega) \to L^2(\Omega;\mathbb{R}^2)$, the *weak gradient*, such that

$$\langle \nabla u, \vec{\varphi} \rangle_{L^2(\Omega;\mathbb{R}^2)} = -\langle u, \operatorname{div} \vec{\varphi} \rangle_{L^2(\Omega)}$$

whenever $\vec{\varphi} \in C_c^1(\Omega;\mathbb{R}^2)$.

11. Consider the regularization functional (1) with

$$\mathcal{R}(u) = \|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^2)} \, . \tag{2}$$

Show that in this case the function $u \in L^2(\Omega)$ is a minimizer of the regularization functional, if and only if it is a weak solution of the partial differential equation

$$\begin{aligned} u - f - \alpha \Delta u &= 0 \quad \text{on } \Omega, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega \, . \end{aligned} \tag{3}$$

Recall that $u \in L^2(\Omega)$ is a weak solution of (3), if and only if $u \in D$ and

$$\langle u, v \rangle_{L^2(\Omega)} + \alpha \langle \nabla u, \nabla v \rangle_{L^2(\Omega;\mathbb{R}^2)} = \langle f, v \rangle_{L^2(\Omega)}$$

for every $v \in D$.

12. Write a Matlab (or Octave) implementation of the linear filter (3). Assume to that end that the domain Ω is rectangular and that the data f are given on a regular grid on Ω. Test the programme for different parameters on the images *Lenna.png*, *Peppers.jpg*, and *Boats.png*. In addition, test the effects of the filter, if noise is added to the images.

You may use either a finite difference or a finite element method for the implementation. In case of finite elements, use either piecewise linear or bilinear basis functions; for finite differences, use a standard 5-point stencil for the discretization of the Laplacian. The method for the solution of the system of linear equations that is easiest to implement might be a Jacobi iteration. You may also use the standard MATLAB solvers, but note that the system matrix will be huge (use sparse matrices!).

You can load images in MATLAB with the `imread` command and display them with `imagesc` (possibly with `colormap(gray)`). Depending on your implementation, it might be necessary to convert the images to `double` at some stage. Random noise can be generated with `rand` (for uniformly distributed random noise in $[0,1]$) and `randn` (for normally distributed noise).

13. The main problem with the linear filter defined by (1) is that it blurs the image heavily. One possibility to mitigate this blurring is to decrease the diffusivity across edges, which can be detected by applying the filter (3) to the noisy, original image f.

    Denote therefore by $u_\alpha \in L^2(\Omega)$ the solution of (3) and define

$$\mathcal{R}(u) = \int_\Omega \frac{|\nabla u(x)|^2}{1 + |\nabla u_\alpha(x)|^2}\, dx \,. \tag{4}$$

    This leads, with a regularization parameter $\beta > 0$, possibly different from α, to the filter

$$\begin{aligned} u - f - \beta \operatorname{div}\Big(\frac{\nabla u}{1 + |\nabla u_\alpha|^2}\Big) &= 0 \quad \text{on } \Omega, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega \,. \end{aligned} \tag{5}$$

    Write a MATLAB or OCTAVE implementation of the filter (5). Compare the results of (5) with the results of (3). You may want to use the discretizations of the gradient and divergence given in Exercise 14 below. In case you need to setup a system matrix, recall the product rule for the divergence,

$$\operatorname{div}\big(c(x)\nabla u(x)\big) = \big\langle \nabla c(x), \nabla u(x)\big\rangle + c(x)\Delta u(x) \,.$$

14. Assume that $\Omega = (0,a) \times (0,b) \subset \mathbb{R}^2$ is a rectangular domain, which we discretize using a uniform grid with nodal points $(ih, jh) \in \Omega$, $1 \le i \le M$, $1 \le j \le N$, for some step size $h > 0$. Consider the discretization of the gradient $\nabla\colon \mathbb{R}^{M\times N} \to (\mathbb{R}^2)^{M\times N}$ by forward differences,

$$(\nabla u)^{(1)}_{i,j} = \begin{cases} \dfrac{u_{i+1,j} - u_{i,j}}{h} & \text{if } i < M\,, \\ 0 & \text{if } i = M\,, \end{cases} \qquad (\nabla u)^{(2)}_{i,j} = \begin{cases} \dfrac{u_{i,j+1} - u_{i,j}}{h} & \text{if } j < N\,, \\ 0 & \text{if } j = N\,. \end{cases}$$

Define moreover the discrete divergence $\operatorname{div}\colon (\mathbb{R}^2)^{M\times N} \to \mathbb{R}^{M\times N}$ as the negative adjoint of ∇, that is,

$$\langle \operatorname{div}\Phi, u\rangle = \langle \Phi, -\nabla u\rangle \qquad \text{whenever } u \in \mathbb{R}^{M\times N} \text{ and } \Phi \in (\mathbb{R}^2)^{M\times N}\,,$$

with $\langle\cdot,\cdot\rangle$ denoting the standard scalar product on Euclidean space. Derive an explicit formula for the discrete divergence.

15. As an alternative to the usage of $|\nabla u_\alpha|^2$ in (4), it is also possible—and more common—to convolve the noisy image f with some smooth convolution kernel ρ_ε, and to use $|\nabla(\rho_\varepsilon * f)|^2$ as an edge detector. More precisely, we assume that $\Omega = (0,a)\times(0,b) \subset \mathbb{R}^2$ is a rectangular domain and consider the following *mirrored* continuation $\tilde f$ of f to $\mathbb{R}^2$ defined, for k, $l \in \mathbb{Z}$, and $0 < x < a$, $0 < y < b$, by

$$\tilde f(ka + x, lb + y) = \begin{cases} f(x,y) & \text{if } k \text{ and } l \text{ are even,}\\ f(a-x,y) & \text{if } k \text{ is odd and } l \text{ is even,}\\ f(x,b-y) & \text{if } k \text{ is even and } l \text{ is odd,}\\ f(a-x,b-y) & \text{if } k \text{ and } l \text{ are odd.}\end{cases}$$

Let therefore $\rho_\varepsilon \in C^\infty(\mathbb{R}^2)$ satisfy $\rho_\varepsilon \ge 0$ and $\int_{\mathbb{R}^2} \rho_\varepsilon(x)\,dx = 1$. Show that the functional $\mathcal{T}_{\alpha,f}(u) = \|u-f\|^2_{L^2(\Omega)} + \alpha\mathcal{R}(u)$ with

$$\mathcal{R}(u) = \int_\Omega \frac{|\nabla u(x)|^2}{1 + |\nabla(\rho_\varepsilon * \tilde f)(x)|^2}\,dx$$

attains a unique minimizer on $L^2(\Omega)$, which is a weak solution of the partial differential equation

$$\begin{aligned} u - f - \alpha \operatorname{div}\left(\frac{\nabla u}{1+|\nabla(\rho_\varepsilon * \tilde f)|^2}\right) &= 0 \quad \text{on } \Omega\,,\\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{6}$$

You may use the fact that under the assumptions above we have that $\rho_\varepsilon * \tilde f \in C^\infty(\mathbb{R}^2)$.

16. Write a MATLAB or OCTAVE implementation of the filter (6) and compare the results with the results of (5). Use as convolution kernel a Gaussian of variance $\varepsilon > 0$.

Note that the function `imfilter` provides a fast implementation of the convolution in the image processing toolbox in MATLAB. Mirrored boundary conditions can be generated by using the option `'symmetric'`.

Image Denoising: Iterative Filtering — Parabolic PDEs

In the linear filter (3), one smoothes the noisy image u by minimizing the regularization functional (1) once, but with a fairly large regularization parameter α. Instead, it is also possible to use a small regularization parameter, but repeatedly, in each step denoising the solution of the previous step further. Then one arrives at the *iterative method*

$$u^{(0)} = f\,, \qquad u^{(k+1)} = \arg\min\bigl\{\|u - u^{(k)}\|^2 + \alpha_k \mathcal{R}_k(u) : u \in L^2(\Omega)\bigr\}\,.$$

Note that, in principle, the regularization parameter α_k and the regularization functional $\mathcal{R}_k$ may change in each iteration step—in particular, $\mathcal{R}_k$ may depend on the outcome $u^{(k)}$ of the previous step. This approach is strongly related to parabolic differential equations and semi-group theory.

17. Prove, using the Hille–Yosida theorem, that the negative Laplace operator $-\Delta$ defines a linear contraction semi-group on $L^2(\Omega)$ (you might want to recall the definitions of Exercise 11).

18. Implement the solution of the parabolic differential equation

$$\begin{aligned} \partial_t u - \Delta u &= 0\,, \\ u(x,0) &= f(x)\,, \quad x \in \Omega\,, \\ \partial_\nu u(x,t) &= 0\,, \qquad x \in \partial\Omega\,,\ t > 0\,, \end{aligned}$$

by computing the iteration

$$u^{(k+1)} = (\mathrm{Id} - \tau\Delta)^{-1} u^{(k)}$$

under the assumption that $\Omega \subset \mathbb{R}^2$ is a rectangular domain and that the intial data f are given on a regular grid on Ω (note that this is by no means efficient: an explicit solution can easily be computed by convolving the data f with a Gaussian kernel of increasing variance).

19. Using the notation of Exercise 15 and fixing some $\varepsilon > 0$, implement the solution of the parabolic differential equation

$$\begin{aligned} \partial_t u - \operatorname{div}\Bigl(\frac{\nabla u}{1 + |\nabla(\rho_\varepsilon * \tilde f)|^2}\Bigr) &= 0\,, \\ u(x,0) &= f(x)\,, \quad x \in \Omega\,, \\ \partial_\nu u(x,t) &= 0\,, \qquad x \in \partial\Omega\,,\ t > 0\,, \end{aligned}$$

again by computing the corresponding iteration.

20. Using the notation of the previous exercises, implement the iteration $u^{(0)} = f$ and

$$\begin{aligned} u^{(k+1)} - \tau \operatorname{div}\Bigl(\frac{\nabla u^{(k+1)}}{1 + |\rho_\varepsilon * \nabla \tilde u^{(k)}|^2}\Bigr) &= u^{(k)}\,, \\ \partial_\nu u^{(k+1)} &= 0\,, \qquad x \in \partial\Omega\,. \end{aligned}$$

Total Variation

The Sobolev space $W^{1,1}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ non-empty, open, and bounded with sufficiently regular boundary (say C^2 or Lipschitz), is defined as the space of all $u \in L^1(\Omega)$ for which there exists a function $\vec{v} \in L^1(\Omega; \mathbb{R}^n)$ such that

$$\int_\Omega \langle \vec{v}(x), \vec{\varphi}(x) \rangle \, dx = -\int_\Omega u(x) \, \operatorname{div} \vec{\varphi}(x) \, dx \tag{7}$$

for all $\vec{\varphi} \in C_c^1(\Omega; \mathbb{R}^n)$. If $u \in W^{1,1}(\Omega)$, then the mapping $\vec{v} \in L^1(\Omega; \mathbb{R}^n)$ satisfying (7) is unique; one usually writes $\vec{v} =: \nabla u$, the weak gradient of u. If $u \in W^{1,1}(\Omega) \cap C^1(\Omega)$, then the Gauß–Green theorem implies that ∇u coincides with the classical gradient of u.

In addition, we define the space $BV(\Omega)$ of functions of *bounded variation* as

$$BV(\Omega) := \left\{ u \in L^1(\Omega) : \sup \left\{ \int_\Omega u \operatorname{div} \vec{\varphi} : \vec{\varphi} \in C_c^1(\Omega; \mathbb{R}^n), \ \|\vec{\varphi}\|_{L^\infty} \leq 1 \right\} < +\infty \right\} .$$

For $u \in BV(\Omega)$ we define the *total variation of* u *on* Ω as

$$|Du|(\Omega) := \sup \left\{ \int_\Omega u \operatorname{div} \vec{\varphi} : \vec{\varphi} \in C_c^1(\Omega; \mathbb{R}^n), \ \|\vec{\varphi}\|_{L^\infty} \leq 1 \right\} .$$

21. Show that $W^{1,1}(\Omega) \subset BV(\Omega)$ and that $|Du|(\Omega) = \int_\Omega |\nabla u(x)| \, dx$ for every $u \in W^{1,1}(\Omega)$.

22. Consider the case $\Omega = (a, b) \subset \mathbb{R}$. Show that every function $u \in W^{1,1}((a, b))$ has a continuous representative.

    Hint: Consider test functions φ *of the form* $\varphi = \rho_\varepsilon * \chi_{(c,d)}$*, where* ρ_ε *is a smooth convolution kernel and* $\chi_{(c,d)}$ *the characteristic function of an interval* $(c, d) \subset (a, b)$*.*

23. Find a function $u \in BV((a, b))$ that has *no* continuous representative.

24. Show that the mapping $\mathcal{R}_1 \colon L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$,

    $$\mathcal{R}_1(u) := \begin{cases} |Du|(\Omega) & \text{if } u \in BV(\Omega) \cap L^2(\Omega) , \\ +\infty & \text{else} \end{cases}$$

    is lower semi-continuous, convex, and positively homogeneous (that is, $\mathcal{R}_1(tu) = |t|\mathcal{R}_1(u)$ for all $u \in L^2(\Omega)$ and $t \in \mathbb{R} \setminus \{0\}$).

25. Consider the mapping $\mathcal{R}_2 \colon L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$,

    $$\mathcal{R}_2(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u(x)| \, dx & \text{if } u \in W^{1,1}(\Omega) \cap L^2(\Omega) , \\ +\infty & \text{else.} \end{cases}$$

    Show that the mapping $\mathcal{R}_2$ is *not* lower semi-continuous on $L^2(\Omega)$.

26. The goal of this exercise is the implementation of an approximation of *total variation filtering*, defined by the minimization of the functional

$$\mathcal{T}_{\alpha,f}(u) := \frac{1}{2}\|u - f\|_{L^2}^2 + \alpha\mathcal{R}_1(u)$$

with $\mathcal{R}_1$ defined as in Exercise 24. The first step is the replacement of the total variation $\mathcal{R}_1$ by the integral over the absolute value of the gradient, i.e., the functional $\mathcal{R}_2$ defined in Exercise 25. The integrand being non-differentiable, it is then replaced by the smooth approximation

$$\mathcal{R}_2^{(\varepsilon)} := \begin{cases} \displaystyle\int_\Omega \sqrt{\varepsilon^2 + |\nabla u(x)|^2}\, dx & \text{if } u \in W^{1,1}(\Omega) \cap L^2(\Omega)\,, \\ +\infty & \text{else,} \end{cases}$$

where $\varepsilon > 0$ is some small parameter. The ensuing functional can then be minimized by solving the non-linear PDE

$$\begin{aligned} u - f - \alpha \operatorname{div}\left(\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}}\right) &= 0 \quad \text{on } \Omega\,, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{8}$$

A common approach for the numerical solution of (8) is the usage of the fixed point iteration

$$\begin{aligned} u^{(k+1)} - f - \alpha \operatorname{div}\left(\frac{\nabla u^{(k+1)}}{\sqrt{\varepsilon^2 + |\nabla u^{(k)}|^2}}\right) &= 0 \quad \text{on } \Omega\,, \\ \partial_\nu u^{(k+1)} &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{9}$$

Write a MATLAB or OCTAVE implementation of the approximation of total variation filtering introduced above using the iteration (9). Compare the results with those of anisotropic diffusion (5).

Mean Curvature Motion

From an axiomatic point of view, one can argue that evolution equations are useful for image processing only if they satisfy certain invariance properties, among them invariance with respect to constrast changes. That is, if $\phi\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing and continuous bijection and u and u^ϕ denote the solutions of the equation with initial values f and $\phi \circ f$, respectively, then we should have that

$$\phi\bigl(u(t,x)\bigr) = u^\phi(t,x)$$

for every t and x. Basically, the only useful evolutions that have this property are governed by equations of the form

$$\partial_t u = |\nabla u|\, G\Bigl(\operatorname{div}\Bigl(\frac{\nabla u}{|\nabla u|}\Bigr)\Bigr),$$

where $G\colon \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function.

27. Write a Matlab or Octave implementation of *mean curvature motion* given by

$$\partial_t u = |\nabla u|\Bigl(\operatorname{div}\Bigl(\frac{\nabla u}{|\nabla u|}\Bigr)\Bigr).$$

Use an explicit Euler method (with sufficiently small step size) for the time discretization, and a simple finite difference method for the space discretization. Moreover, regularize the denominator like for total variation regularization replacing $|\nabla u|$ by $\sqrt{\varepsilon^2 + |\nabla u|^2}$. If one uses a non-symmetric space discretization, it can be useful to rotate the image by 90 degrees using `rot90` after each time step in order to minimize artifacts.

Test the programme on the images *Cards.png* and *Lenna.png*. In addition, verify that the evolution indeed is (approximately) invariant with respect to gray-value transformations by applying different contrast changes to the initial function and comparing the respective results.

28. The same as Exercise 27, but with *affine invariant mean curvature motion* given by

$$\partial_t u = |\nabla u|\Bigl(\operatorname{div}\Bigl(\frac{\nabla u}{|\nabla u|}\Bigr)\Bigr)^{1/3}.$$

Here, in theory, the evolution should also be invariant with respect to affine transformations of the domain of the image.

29. Verify numerically that neither the linear filter of Exercise 18 nor total variation flow given by

$$\partial_t u = \Bigl(\operatorname{div}\Bigl(\frac{\nabla u}{|\nabla u|}\Bigr)\Bigr)$$

are contrast invariant.