An Introduction to Signal and Image Processing

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Last update on June 14th, 2012

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Chapter 1

Fourier Transform

Many phenomena are periodic in time. Examples are *amperage* and *voltage* of *alternate current* (AC). Mathematically, such quantities are described by periodic functions.

Definition 1.1. A function f is called *periodic* if there exists a number $T \in \mathbb{R}$, called *period*, such that for all $t \in \mathbb{R}$

$$f(t+T) = f(t) \ . \qquad \blacksquare$$

In engineering often the term frequency, is used when periodic functions are involved, which is closely related to the period of the function.

Definition 1.2. Let f be a periodic function with period $T \in \mathbb{R}$. Then $1/T$ is called the *frequency* of f. The unit of measurement of frequency is [Hz] (Hertz):

$$1s^{-1} = 1\text{Hz} \ . \qquad \blacksquare$$

To become more familar with the notion of periodicity we consider next the movement of a particle on a circle with arbitrary radius $r > 0$.

Definition 1.3. Let $\phi(t) \in (-\pi, \pi]$ denote the *angular position* of a particle on a circle with arbitrary radius $r > 0$ at time t, i.e.,

$$\Big(\sin\big(\phi(t)\big), \cos\big(\phi(t)\big)\Big) \in \mathbb{S}^1 \subset \mathbb{R}^2 \ .$$

The *angular velocity* ν of a moving particle is the ratio of the variation of the angle, $\Delta\phi$, relative to variation of time, Δt, i.e.,

$$\nu := \frac{\Delta\phi}{\Delta t} \, .$$

In the limit, differential calculus shows that

$$\nu(t) = \phi'(t) \, , \tag{1.1}$$

where ϕ' denotes the derivative of ϕ. ∎

Definition 1.4. Let $T \in \mathbb{R}$ denote the required time of a particle with constant angular velocity on a circle for a full cycle. Then

$$\omega := \frac{2\pi}{T}$$

is called *angular frequency*. ∎

The movement of the particle, seen as a function of time, requires $T \in \mathbb{R}$ time units for a full cycle (equal to the period). Assuming a uniform motion of the particle with constant velocity v we obtain

$$T = \frac{2\pi r}{v} \, . \tag{1.2}$$

The angular velocity ν according to v is given by

$$\nu = \frac{v}{r} \, .$$

From (1.2) we infer that

$$\frac{2\pi}{\nu} = \frac{2\pi r}{v} = T \, .$$

Remark 1.5. Notice that in case of a movement on a circle the term angular frequency is somehow missleading: it is a re-scaled frequency ($2\pi/T$ in contrast to $1/T$) and coincides with the mean angular velocity. ∎

Example 1.6. A typical example of a non-constant, periodic function is given by

$$t \to A \sin(\omega t + \alpha) \, ,$$

where $|A|$ denotes the *amplitude* and α the *phase shift*. As the period of this function equals $2\pi/\omega$, we see that ω equals the angular frequency. ∎

1.1 Trigonometric Polynomials

Trigonometric polynomials are used to approximate periodic functions. First we introduce common abbreviations for functions used throughout this lecture notes:

Definition 1.7. For $k \in \mathbb{Z}$ we define

$$\begin{aligned} \exp_k &: \mathbb{R} \to \mathbb{C}\,, \quad \theta \mapsto \exp(\mathtt{i}k\theta)\,, \\ \sin_k &: \mathbb{R} \to \mathbb{R}\,, \quad \theta \mapsto \sin(k\theta)\,, \\ \cos_k &: \mathbb{R} \to \mathbb{R}\,, \quad \theta \mapsto \cos(k\theta)\,. \end{aligned}$$

■

Definition 1.8. For $n \in \mathbb{N}_0$ let

$$P_{T[n]} := \mathtt{span}\{\exp_k : -n \leq k \leq n\}$$

be the set of linear combinations of functions $\exp_k$, $-n \leq k \leq n$. Every element $p \in P_{T[n]}$,

$$p(\theta) = \sum_{k=-n}^{n} \alpha_k \exp(\mathtt{i}k\theta)\,, \quad \alpha_k \in \mathbb{C}\,, \tag{1.3}$$

is called *trigonometric polynomial of degree n*. ■

We recall that for $\theta \in (-\pi, \pi)$ and $k, l \in \mathbb{N}$ the functions $\sin_k$ and $\cos_k$ are orthogonal with respect to the inner product of $L^2((-\pi, \pi); \mathbb{R})$, that is, for all $k, l \in \mathbb{N}$

$$\int_{-\pi}^{\pi} \sin(k\theta)\cos(l\theta)\; d\theta = 0\;,$$

and, in addition, for all $k, l \in \mathbb{N}$ with $k \neq l$

$$\int_{-\pi}^{\pi} \sin(k\theta)\sin(l\theta)\; d\theta = 0 = \int_{-\pi}^{\pi} \cos(k\theta)\cos(l\theta)\; d\theta\;.$$

From the orthogonality relations it follows that the set of functions

$$\{\cos_k, \sin_k : k \in \mathbb{N}\}$$

consists of linearly independent functions. Therefore, the same is true for the set of the respective periodic extensions of these functions to $\mathbb{R}$.

Let $p \in P_{T[n]}$ with coefficients

$$\alpha_k = \frac{a_k}{2} - \mathtt{i}\frac{b_k}{2}\,, \quad a_k \in \mathbb{R},\ b_k \in \mathbb{R}, \quad -n \le k \le n\ .$$

Then, since $\exp(\mathtt{i}k\theta) = \cos(k\theta) + \mathtt{i}\sin(k\theta)$, the trigonometric polynomial p of degree n can be written as

$$\begin{aligned} p(\theta) = & \sum_{k=-n}^{n} \left(\frac{a_k}{2}\cos(k\theta) + \frac{b_k}{2}\sin(k\theta)\right) + \\ & \mathtt{i}\sum_{k=-n}^{n} \left(-\frac{b_k}{2}\cos(k\theta) + \frac{a_k}{2}\sin(k\theta)\right)\ . \end{aligned}$$

The linear independence of the sine and cosine functions occurring in the imaginary part of p implies that the trigonometric polynomial of degree n is real valued if and only if

$$\sum_{k=-n}^{n} b_k\cos(k\theta) = 0 \quad \text{and} \quad \sum_{k=-n}^{n} a_k\sin(k\theta) = 0\ .$$

This is the case if and only if $b_k = -b_{-k}$ and $a_k = a_{-k}$, or equivalently,

$$\alpha_k = \bar{\alpha}_{-k} \text{ for } 0 \le k \le n\ ,$$

where $\bar{\alpha}_{-k}$ denotes the complex conjugate of α_{-k}. In particular we see that

$$b_0 = 0\ .$$

Thus

$$\begin{aligned} p(\theta) &= \sum_{k=-n}^{n} \left(\frac{a_k}{2}\cos(k\theta) + \frac{b_k}{2}\sin(k\theta)\right) \\ &= \frac{a_0}{2} + \sum_{k=-n}^{-1} \left(\frac{a_k}{2}\cos(k\theta) + \frac{b_k}{2}\sin(k\theta)\right) \\ &\quad + \sum_{k=1}^{n} \left(\frac{a_k}{2}\cos(k\theta) + \frac{b_k}{2}\sin(k\theta)\right) \end{aligned}$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n}\left(\frac{a_{-k}}{2}\cos(-k\theta) + \frac{b_{-k}}{2}\sin(-k\theta)\right) \\ + \sum_{k=1}^{n}\left(\frac{a_k}{2}\cos(k\theta) + \frac{b_k}{2}\sin(k\theta)\right) .$$

That is, a *real valued trigonometric polynomial of degree* n can be represented by

$$p(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(k\theta) + \sum_{k=1}^{n} b_k \sin(k\theta) , \tag{1.4}$$

with

$$\begin{aligned} a_0 &= 2\alpha_0 , \\ a_k &= 2\,\mathrm{Re}(\alpha_k) , \quad 1 \le k \le n , \\ b_k &= -2\,\mathrm{Im}(\alpha_k) , \quad 1 \le k \le n . \end{aligned}$$

The theorem below states that each square integrable (complex valued) function f defined on $(-\pi, \pi)$ can be approximated by trigonometric polynomials. For proving this theorem we need the following lemma:

Lemma 1.9. *Let* $\{\phi_n : n \in \mathbb{N}\}$ *be a countable, complete orthonormal basis of the Hilbert space* X*. Denote by* $\langle\, , \rangle$ *the inner product and by* $\|\cdot\|$ *the corresponding norm on* X*. Then, for every* $f \in X$ *and* $N \in \mathbb{N}$

$$f_N = \sum_{n=1}^{N} \alpha_n \phi_n , \quad \alpha_n := \langle f, \phi_n \rangle \tag{1.5}$$

is the best approximation of f *on the set*

$$X_N := \mathtt{span}\,\{\phi_n : n = 1, \ldots, N\} .$$

More precisely, for every $g \in X_N$ *with* $g \neq f_N$ *we have*

$$\|f - g\| > \|f - f_N\| .$$

Proof. We use elementary identities of inner products on Hilbert spaces:

For every $g = \sum_{n=1}^{N} \beta_n \phi_n \in X_N$ with $\beta_n = \langle g, \phi_n \rangle$ we have

$$\begin{aligned} \|f-g\|^2 &= \langle f-g, f-g \rangle \\ &= \langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + \langle g, g \rangle \\ &= \|f\|^2 - \sum_{n=1}^{N} (\alpha_n \bar{\beta}_n + \beta_n \bar{\alpha}_n) + \sum_{n=1}^{N} |\beta_n|^2 \\ &= \|f\|^2 - \sum_{n=1}^{N} |\alpha_n|^2 + \sum_{n=1}^{N} |\beta_n - \alpha_n|^2 \,. \end{aligned}$$

The right hand side of this identity is minimal if and only if $\beta_n = \alpha_n$ for $n = 1, \ldots, N$. In that case f_N and g coincide, which shows that f_N is the best approximation of f on X_N. □

Theorem 1.10. *The family of functions*

$$\frac{1}{\sqrt{2\pi}} \exp_k \,, \quad -n \le k \le n \,, \tag{1.6}$$

is an orthonormal basis on $P_{T[n]}$ with respect to the inner product of the Hilbert space $L^2((-\pi, \pi); \mathbb{C})$.

The best approximation of a function $f \in L^2((-\pi, \pi); \mathbb{C})$ on $P_{T[n]}$ with respect to the $L^2((-\pi, \pi); \mathbb{C})$–norm is

$$f_b(\theta) = \sum_{k=-n}^{n} \alpha_k \exp(\mathtt{i} k \theta)$$

with coefficients

$$\alpha_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \exp(-\mathtt{i} k \theta) \, d\theta \,, \quad -n \le k \le n \,. \tag{1.7}$$

Proof. First, note that the inner product of $L^2((-\pi, \pi); \mathbb{C})$ is defined as

$$\langle f, g \rangle_{L^2((-\pi,\pi);\mathbb{C})} := \int_{-\pi}^{\pi} f(\theta) \bar{g}(\theta) \, d\theta \,.$$

Thus, for $-n \leq j, k \leq n$ we have

$$\frac{1}{2\pi}\langle \exp_k, \exp_j\rangle_{L^2((-\pi,\pi);\mathbb{C})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(\mathtt{i}k\theta)\exp(-\mathtt{i}j\theta)\,d\theta$$
$$= \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} d\theta = 1\,, & \text{if } k = j\,, \\ \dfrac{1}{2\pi}\dfrac{1}{\mathtt{i}(k-j)}\exp\big(\mathtt{i}(k-j)\theta\big)\Big|_{-\pi}^{\pi} = 0\,, & \text{if } k \neq j\,. \end{cases}$$

This proves the first assertion. The second part is an immediate consequence of Lemma 1.9. □

Definition 1.11. Let $f \in L^2((-\pi,\pi);\mathbb{C})$ and $k \in \mathbb{Z}$. The terms

$$\alpha_k := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\exp(-\mathtt{i}k\theta)d\theta \tag{1.8}$$

are called *Fourier coefficients* of the function f. ■

Notice that the coefficients α_k of the best approximation are independent of n. Therefore, by taking the limit $n \to \infty$ the sequence of best approximations converges formally to a series.

Definition 1.12. Let $f \in L^2((-\pi,\pi);\mathbb{C})$ and α_k, $k \in \mathbb{Z}$, its Fourier coefficients defined in (1.8). The series defined by

$$\sum_{k=-\infty}^{\infty} \alpha_k \exp_k := \lim_{n\to\infty}\sum_{k=-n}^{n} \alpha_k \exp_k \tag{1.9}$$

is called *Fourier series* of f. ■

Remark 1.13. The Fourier series of a function $f \in L^2((-\pi,\pi);\mathbb{C})$ is usually written as

$$f = \sum_{k=-\infty}^{\infty} \alpha_k \exp_k\ . \tag{1.10}$$

WARNING: In this notation the argument is suppressed, because otherwise one could think of pointwise convergence of this series to the function f, which is not guaranteed. In fact, one can show that the formal Fourier series of $f \in L^2((-\pi,\pi);\mathbb{C})$ converges with respect to the L^2–norm to the function f, but convergence in the quadratic mean does not imply pointwise convergence in general! We refer to Remark 1.16 for details on pointwise and uniform convergence of the Fourier series. ■

For the convergence of the Fourier series in the L^2–sense the following characterization of the space $L^2((-\pi,\pi);\mathbb{C})$ is essential:

Theorem 1.14. *Let f be integrable. Then $f \in L^2((-\pi,\pi);\mathbb{C})$ if and only if $\sum_{k=-\infty}^{\infty} |\alpha_k|^2 < \infty$.*

Next we estimate the error of f to its best approximating trigonometric polynomial f_b with respect to the L^2–norm, showing that L^2–convergence of the Fourier series to f is guaranteed (cf. Remark 1.13). Using that $1/\sqrt{2\pi}\,\mathrm{exp}_k$, $-\infty \leq k \leq \infty$ is an orthonormal system on $L^2((-\pi,\pi);\mathbb{C})$, it follows that

$$\|f - f_b\|^2_{L^2((-\pi,\pi);\mathbb{C})} = \sum_{|k|>n} |\alpha_k|^2 \int_{-\pi}^{\pi} \exp(\mathtt{i}k\theta)\exp(-\mathtt{i}k\theta)\,d\theta = 2\pi \sum_{|k|>n} |\alpha_k|^2 \ . \tag{1.11}$$

In addition, in the sequel of this lecture notes we sometimes assume that we can do some calculations with the formal Fourier series, such as, e.g., differentiation:

$$f' = \mathtt{i} \sum_{k=-\infty}^{\infty} k\alpha_k \,\mathrm{exp}_k \ . \tag{1.12}$$

Lemma 1.15. *Define*

$$B := \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos_k, \frac{1}{\sqrt{\pi}}\sin_k : 1 \leq k \leq n \right\}$$

and

$$P^{\mathbb{R}}_{T[n]} := \mathtt{span}\{b : b \in B\}\,.$$

The elements $b \in B$ are orthonormal with respect to the inner product of the space $L^2((-\pi,\pi);\mathbb{R})$. The best approximation f_b of a function $f \in L^2((-\pi,\pi);\mathbb{R})$ on $P^{\mathbb{R}}_{T[n]}$ with respect to the $L^2((-\pi,\pi);\mathbb{R})$–norm is given by

$$\begin{aligned} f_b(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\,d\tau &+ \frac{1}{\pi}\sum_{k=1}^{n}\left(\int_{-\pi}^{\pi} f(\tau)\cos(k\tau)d\tau\right)\cos(k\theta) \\ &+ \frac{1}{\pi}\sum_{k=1}^{n}\left(\int_{-\pi}^{\pi} f(\tau)\sin(k\tau)d\tau\right)\sin(k\theta)\ . \end{aligned}$$

If f_b is viewed as an element of the space $P_{T[n]}$ the corresponding Fourier coefficients $\alpha_k = a_k/2 - \mathtt{i} b_k/2$ satisfy

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(k\theta) d\theta\,, \quad 1 \le k \le n\,, \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(k\theta) d\theta\,, \quad 1 \le k \le n\;.
\end{aligned}
$$

Fourier analysis is concerned with the analysis of convergence of the Fourier series. In the sequel we give three basic results on convergence of the Fourier series.

Remark 1.16. The following theorems and the corresponding proofs can be found in [4]:

- Let $f \in L^2((-\pi,\pi);\mathbb{C})$ and 2π–periodic. Assume that at the point θ the four limits [1]

  $$f(\theta_+), \quad f(\theta_-), \quad \lim_{t\to 0+} \frac{f(\theta+t) - f(\theta_+)}{t}, \quad \lim_{t\to 0+} \frac{f(\theta-t) - f(\theta_-)}{t},$$

  exist. Then the Fourier series of f at the point θ converges to

  $$\frac{1}{2}\big(f(\theta_+) + f(\theta_-)\big)\;.$$

- Let f be continuous on $\mathbb{R}$ and 2π–periodic. We denote by f_n the best approximation on $P_{T[n]}$ of $f|_{(-\pi,\pi)}$, i.e, the approximation by the finite sum (of order n) based on the Fourier coefficients of f (cf. Theorem 1.10). Then

  $$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N} f_n$$

  converges uniformly on $[-\pi,\pi]$ to f.

- Let f be 2π–periodic and piecewise continuously differentiable on the interval $[-\pi,\pi]$. Then its Fourier series converges uniformly to f on every compact interval where f is continuous. ■

[1] $f(\theta_+) = \lim_{h>0\to 0} f(\theta+h)$ and $f(\theta_-) = \lim_{h<0\to 0} f(\theta-h)$.

Example 1.17. We calculate the Fourier series of the characteristic function $f := \chi_{[a,b]}$ of the interval $[a,b] \subseteq [-\pi,\pi]$. The Fourier coefficients of $\chi_{[a,b]}$ are given by

$$\alpha_k = \frac{1}{2\pi}\int_a^b \exp(-\mathtt{i}k\theta)\,d\theta\,, \quad k \in \mathbb{Z}\,.$$

Define

$$c := (a+b)/2\,, \quad d := (b-a)/2\,.$$

Thus, $c+d=b$ and $c-d=a$. Then for $k=0$ we obtain

$$\alpha_0 = (b-a)/2\pi = d/\pi$$

and for $k \neq 0$

$$\begin{aligned}\alpha_k &= \frac{1}{2\pi}\frac{1}{-\mathtt{i}k}\exp(-\mathtt{i}k\theta)\Big|_{\theta=a}^{\theta=b}\\ &= -\frac{1}{2k\pi\mathtt{i}}\exp(-\mathtt{i}kc)\big(\exp(-\mathtt{i}kd)-\exp(\mathtt{i}kd)\big)\\ &= -\frac{1}{2k\pi\mathtt{i}}\exp(-\mathtt{i}kc)\big(\cos(kd)-\mathtt{i}\sin(kd)-\cos(kd)-\mathtt{i}\sin(kd)\big)\\ &= \frac{1}{\pi}\exp(-\mathtt{i}kc)\frac{\sin(kd)}{k}\,.\end{aligned}$$

Thus, the Fourier series of $\chi_{[a,b]}$ is given by

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\alpha_k\exp(\mathtt{i}k\theta) &= \frac{d}{\pi}+\frac{1}{\pi}\sum_{k\in\mathbb{Z}\setminus\{0\}}\exp(-\mathtt{i}kc)\frac{\sin(kd)}{k}\exp(\mathtt{i}k\theta)\\ &= \frac{d}{\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(kd)}{k}\Big(\exp\big(\mathtt{i}k(\theta-c)\big)+\exp\big(-\mathtt{i}k(\theta-c)\big)\Big)\\ &= \frac{d}{\pi}+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin(kd)}{k}\cos\big(k(\theta-c)\big)\,.\end{aligned}$$

Let f_b be the best approximation of $\chi_{[a,b]}$ in $P_{T[n]}$. Then, by taking into account that $|\alpha_k| = |\alpha_{-k}|$, it follows from (1.11) that

$$\|f-f_b\|^2_{L^2((-\pi,\pi);\mathbb{C})} = 4\pi\sum_{k=n+1}^{\infty}|\alpha_k|^2 = \frac{4}{\pi}\sum_{k=n+1}^{\infty}\frac{\sin^2(kd)}{k^2}\,.$$

The right hand side can be estimated by

$$\frac{4}{\pi}\sum_{k=n+1}^{\infty} k^{-2} \approx \frac{4}{\pi}\int_n^{\infty} t^{-2}\,dt = \frac{4}{\pi}n^{-1} = \mathcal{O}(1/n)\ . \qquad \blacksquare$$

Real Sine and Cosine Expansions

In this section we show that depending on the function f to be investigated it is possible to reduce the Fourier expansion of f to cosine or sine terms, respectively.

Definition 1.18. A function f is called *odd* if for all values θ in the domain

$$f(\theta) = -f(-\theta)\,.$$

Similarly, a function g is called *even* if for all values θ in the domain

$$g(\theta) = g(-\theta)\,. \qquad \blacksquare$$

In the following we assume $f, g \in L^2((-\pi,\pi);\mathbb{R})$ to be 2π–periodic. First we study the case of an odd function f. We observe that

$$\int_{-\pi}^{\pi} f(\tau)\,d\tau = 0$$

and, in addition, as $f\cos_k$ is odd too, for all $k \in \mathbb{N}$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\cos(k\tau)\,d\tau = 0.$$

As a consequence, the Fourier series expansion of an odd function only consists of sine terms:

$$f = \sum_{k=1}^{\infty} b_k \sin_k\,,$$

where

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(\tau)\sin(k\tau)\,d\tau\,.$$

In the last identity we used the fact that for an odd function f the term $f \sin_k$ is an even function, and therefore

$$\int_{-\pi}^{\pi} f(\tau)\sin(k\tau)\,d\tau = 2\int_{0}^{\pi} f(\tau)\sin(k\tau)\,d\tau\,.$$

Now we consider the case of an even function g. Similarly to the case of an odd function, we see that $g\cos_k$ is even too, and $g\sin_k$ is odd. Thus, we obtain

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\tau)\cos(k\tau)\,d\tau = \frac{2}{\pi}\int_{0}^{\pi} g(\tau)\cos(k\tau)\,d\tau.$$

and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\tau)\sin(k\tau)\,d\tau = 0\,.$$

As a consequence, the Fourier series expansion of an even function only consists of cosine terms:

$$g = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos_k\,,$$

where

$$a_k = \frac{2}{\pi}\int_{0}^{\pi} g(\tau)\cos(k\tau)\,d\tau\,.$$

In order to expand a function f defined on the interval $(0,\pi)$ into a Fourier series, f has to be extended to a quadratically integrable function on $(-\pi,\pi)$ (and, subsequently, may be expanded to a 2π- periodic function on $\mathbb{R}$). Denoting the extended 2π-periodic function by g we know that it coincide with the original function f on $(0,\pi)$. Depending on whether f was extended even or odd onto $(-\pi,\pi)$ the Fourier expansion of g only consists of cosine or sine terms.

1.2 Sobolev Spaces

$H^1((-\pi,\pi);\mathbb{C})$ is the space of periodic, square integrable functions, with square integrable (weak) first derivative[2]. With $f, g \in H^1((-\pi,\pi);\mathbb{C})$ we

[2] *Weak* derivative just refers to the fact that it is not a continuous function.

associate the inner product

$$\langle f, g\rangle_{H^1((-\pi,\pi);\mathbb{C})} := \int_{-\pi}^{\pi} f(\tau)\bar{g}(\tau)\, d\tau + \int_{-\pi}^{\pi} f'(\tau)\bar{g}'(\tau)\, d\tau$$

and the norm

$$\|f\|^2_{H^1((-\pi,\pi);\mathbb{C})} := \langle f, f\rangle^2_{H^1((-\pi,\pi);\mathbb{C})} = \|f\|^2_{L^2((-\pi,\pi);\mathbb{C})} + \|f'\|^2_{L^2((-\pi,\pi);\mathbb{C})} \ .$$

For every function $f \in H^1((-\pi,\pi);\mathbb{C})$ with Fourier coefficients α_k, $k \in \mathbb{Z}$, there exists

$$f' = \sum_{k=-\infty}^{\infty} \beta_k \exp_k = \sum_{k=-\infty}^{\infty} \mathtt{i}k\alpha_k \exp_k \tag{1.13}$$

satisfying

$$f(\theta) = f(-\pi) + \int_{-\pi}^{\theta} f'(t)\, dt \ .$$

Since f is 2π-periodic it follows that

$$f(-\pi) = f(\pi) = f(-\pi) + \int_{-\pi}^{\pi} f'(t)\, dt \ .$$

This implies that

$$\beta_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)\, d\theta = 0 \ .$$

Remark 1.19. In 1D every function f belonging to $H^1((-\pi,\pi);\mathbb{C})$ is bounded and continuous. Thus, one can evaluate f at a specific point. ■

Below, we give a complete characterization of the functions in the Sobolev space $H^1((-\pi,\pi);\mathbb{C})$.

Theorem 1.20. *Let $f \in L^2((-\pi,\pi);\mathbb{C})$ with Fourier coefficients $\{\alpha_k\}$. Then $f \in H^1((-\pi,\pi);\mathbb{C})$ if and only if*

$$\sum_{k=-\infty}^{\infty} k^2 |\alpha_k|^2 < \infty \ .$$

Moreover,

$$\frac{1}{2\pi}\|f\|^2_{H^1((-\pi,\pi);\mathbb{C})} = \sum_{k=-\infty}^{\infty} (k^2+1)\, |\alpha_k|^2 \ .$$

Proof. For $f \in H^1((-\pi,\pi);\mathbb{C})$ the Fourier coefficients of the derivative $f' \in L^2((-\pi,\pi);\mathbb{C})$ are given by $\mathtt{i}k\alpha_k$. From Theorem 1.14 it follows that

$$\sum_{k=-\infty}^{\infty} k^2|\alpha_k|^2$$

is convergent and according to the proof of Theorem 1.10 (criterion of orthogonality) we see that

$$\begin{aligned}\sum_{k=-\infty}^{\infty}(k^2+1)\,|\alpha_k|^2 &= \sum_{k=-\infty}^{\infty}k^2\,|\alpha_k|^2+\sum_{k=-\infty}^{\infty}|\alpha_k|^2\\ &=\frac{1}{2\pi}\left(\|f'\|^2_{L^2((-\pi,\pi);\mathbb{C})}+\|f\|^2_{L^2((-\pi,\pi);\mathbb{C})}\right)\\ &=\frac{1}{2\pi}\|f\|^2_{H^1((-\pi,\pi);\mathbb{C})}\,.\end{aligned}$$

Now assume that $\sum_{k=-\infty}^{\infty}k^2\,|\alpha_k|^2$ is convergent. Then, according to Theorem 1.14 the function

$$g=\sum_{k=-\infty}^{\infty}\mathtt{i}k\alpha_k\exp_k$$

is an element of $L^2((-\pi,\pi);\mathbb{C})$. The antiderivative G of g is absolutely continuous and, consequently, an element of $L^2((-\pi,\pi);\mathbb{C})$ too:

$$G=\sum_{k=-\infty}^{\infty}\beta_k\exp_k\,,\quad \beta_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}G(\theta)\exp(-\mathtt{i}k\theta)\,d\theta\,.$$

Thus, using partial integration we obtain for $k\neq 0$

$$\begin{aligned}2\pi\beta_k &= \int_{-\pi}^{\pi}G(\theta)\exp(-\mathtt{i}k\theta)\,d\theta=\frac{\exp(-\mathtt{i}k\theta)}{-\mathtt{i}k}G(\theta)\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\frac{\exp(-\mathtt{i}k\theta)}{-\mathtt{i}k}g(\theta)\,d\theta\\ &=\frac{1}{\mathtt{i}k}\int_{-\pi}^{\pi}\sum_{l=-\infty}^{\infty}\mathtt{i}l\alpha_l\exp(\mathtt{i}l\theta)\exp(-\mathtt{i}k\theta)\,d\theta\\ &=\frac{1}{\mathtt{i}k}\sum_{l=-\infty}^{\infty}\mathtt{i}l\alpha_l\int_{-\pi}^{\pi}\exp(\mathtt{i}l\theta)\exp(-\mathtt{i}k\theta)\,d\theta\\ &=\frac{2\pi\mathtt{i}k\alpha_k}{\mathtt{i}k}=2\pi\alpha_k\,.\end{aligned}$$

Therefore, the Fourier coefficients of f and G coincide for $k \neq 0$, which implies that G equals f up to a constant. This shows that f is differentiable with weak derivative g, i.e., $f \in H^1((-\pi, \pi); \mathbb{C})$. □

Definition 1.21. Let $s > 0$. The function space

$$H^s((-\pi,\pi);\mathbb{C}) := \left\{ f \in L^2((-\pi,\pi);\mathbb{C}) : f = \sum_{k=-\infty}^{\infty} \alpha_k \exp_k, \sum_{k=-\infty}^{\infty} |k|^{2s} |\alpha_k|^2 < \infty \right\}$$

is called the *Sobolev space of periodic functions of order s*. ■

Theorem 1.22. *Let $f, g \in H^s((-\pi,\pi);\mathbb{C})$, $s > 0$, with Fourier coefficients $\{\alpha_k\}, \{\beta_k\}$, respectively. Then*

$$\langle f, g \rangle_{H^s((-\pi,\pi);\mathbb{C})} := 2\pi \sum_{k=-\infty}^{\infty} (|k|^{2s} + 1)\alpha_k \bar{\beta}_k$$

is an inner product on $H^s((-\pi,\pi);\mathbb{C})$.

Remark 1.23. Below we summarize some basic properties of Sobolev spaces of periodic functions:

1. For $0 < s < r$ we have

   $$H^r((-\pi,\pi);\mathbb{C}) \subseteq H^s((-\pi,\pi);\mathbb{C}) .$$

   In particular, for $s > 1$ we see that

   $$H^s((-\pi,\pi);\mathbb{C}) \subseteq H^1((-\pi,\pi);\mathbb{C}) ,$$

   which shows that every function $f \in H^s((-\pi,\pi);\mathbb{C})$, $s \geq 1$, has a weak derivative $f' \in H^{s-1}((-\pi,\pi);\mathbb{C})$. More general, every function $f \in H^s((-\pi,\pi);\mathbb{C})$, has $\lfloor s \rfloor$ weak derivatives, i.e.,

   $$f', f'', \ldots, f^{\lfloor s \rfloor} \in L^2((-\pi,\pi);\mathbb{C}) .$$

2. For $f \in H^s((-\pi,\pi);\mathbb{C})$, $s > 1/2$, its Fourier series converges uniformly on $[-\pi, \pi]$ and, in addition, f is continuous.

3. Every function $f \in H^1((-\pi,\pi);\mathbb{C})$ is Hölder-continuous with constant $C = \|f\|_{H^1((-\pi,\pi);\mathbb{C})}$ and exponent $\alpha = 1/2$, i.e., for all $\theta_1, \theta_2 \in (-\pi,\pi)$

$$|f(\theta_1) - f(\theta_2)| \leq \|f\|_{H^1((-\pi,\pi);\mathbb{C})} |\theta_1 - \theta_2|^{1/2} . \qquad \blacksquare$$

Proof. Because $f \in L^2((-\pi,\pi);\mathbb{C})$ we have the identity

$$f(\theta) = \sum_{k=-\infty}^{\infty} \alpha_k \exp(\mathtt{i}k\theta) \text{ almost everywhere.}$$

Let

$$f_n(\theta) = \sum_{|k|\leq n} \alpha_k \exp(\mathtt{i}k\theta) \text{ (everywhere!)} .$$

ad 2.: Using Cauchy-Schwarz inequality we get for almost all $\theta \in [-\pi,\pi]$ that

$$\begin{aligned}
\left| f(\theta) - f_n(\theta) \right| &\leq \sum_{|k|=n+1}^{\infty} |\alpha_k \exp(\mathtt{i}k\theta)| = \sum_{|k|=n+1}^{\infty} k^{-s}(k^s |\alpha_k|) \\
&\leq \left(\sum_{|k|=n+1}^{\infty} k^{-2s} \right)^{1/2} \left(\sum_{|k|=n+1}^{\infty} k^{2s} |\alpha_k|^2 \right)^{1/2} \\
&\leq \frac{1}{\sqrt{2\pi}} \|f\|_{H^s((-\pi,\pi);\mathbb{C})} \left(\sum_{|k|=n+1}^{\infty} k^{-2s} \right)^{1/2} .
\end{aligned}$$

Since the functions f_n are continuous on $[-\pi,\pi]$ and converge uniformly, the limiting function f is continuous as well, and thus the above inequality holds everywhere.

Since $(\sum_{k=0}^{\infty} k^{-2s})^{1/2}$ is convergent, $\left(\sum_{|k|=n+1}^{\infty} k^{-2s}\right)^{1/2} \to 0$ for $n \to +\infty$, and the assertion follows. $\square$

1.3 Trigonometric Interpolation

We study the numerical approximation of a periodic function f by trigonometric polynomials. In practice the Fourier coefficients have to be approximated numerically on a computer system. For the approximation of the coefficients used in the trigonometric polynomial it is common to use the *trapezoidal* quadrature formula. The resulting polynomial is called trigonometric interpolation polynomial to indicate the use of the approximating coefficients. The reason for the name affix *interpolation* will become more clear in the sequel of this section (cf. Theorem 1.32).

First for $N \in \mathbb{N}$ we define $h := 2\pi/N$ and the knots on the interval $[-\pi, \pi)$

$$\theta_\nu := \nu h - \pi \,, \quad 0 \le \nu \le N-1 \,. \tag{1.14}$$

We assume f to be 2π–periodic. Then we get from the trapezoidal formula the approximation

$$\begin{aligned}\alpha_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\exp(-\mathtt{i}k\theta)\,d\theta \\ &\approx \frac{1}{N}\sum_{\nu=0}^{N-1} f(\theta_\nu)\exp(-\mathtt{i}k\theta_\nu)\,.\end{aligned}$$

Definition 1.24. Let $f \in L^2((-\pi,\pi);\mathbb{C})$, $N \in \mathbb{N}$ and θ_ν, $0 \le \nu \le N-1$ as defined in (1.14). Then for $k \in \mathbb{Z}$

$$\hat{\alpha}_k := \frac{1}{N}\sum_{\nu=0}^{N-1} f(\theta_\nu)\exp(-\mathtt{i}k\theta_\nu) \tag{1.15}$$

is called the k^{th} *approximated Fourier coefficient of f.* ■

We introduce now *interpolating trigonometric polynomials* and justify this name in the following.

Definition 1.25. Let $n, N \in \mathbb{N}$ with $n \le N/2$. Assume $f \in H^s((-\pi,\pi);\mathbb{C})$, $s > 1/2$, with approximating Fourier coefficients $\hat{\alpha}_k$. Then the trigonometric polynomials defined as

$$\begin{aligned}\hat{p}_n(\theta) &:= \sum_{k=-n}^{n} \hat{\alpha}_k \exp(\mathtt{i}k\theta) \quad \text{if } n < N/2\,, \\ \hat{p}_n(\theta) &:= \sum_{k=1-n}^{n} \hat{\alpha}_k \exp(\mathtt{i}k\theta) \quad \text{if } n = N/2\,.\end{aligned} \tag{1.16}$$

are called *interpolating trigonometric polynomials of order n.* ■

Remark 1.26. The interpolating trigonometric polynomial in 1.16 can have at most N coefficients; in the first case the polynomial has $2n+1$ coefficients and in the second case exactly $N = 2n$ coefficients. In the following we prove that the trigonometric interpolation polynomial of order n not only provides the best approximation of a function f with respect to the discrete inner product $\langle\!\langle\, ,\rangle\!\rangle$ but also, if n is proper choosen, interpolates given function values $f(\theta_\nu)$ for all $\nu = 0, 1, \ldots, N-1$. ∎

For deriving the interpolation property it is convenient to introduce the following discrete inner product.

Definition 1.27. Let $N \in \mathbb{N}$ and $s > 1/2$. For $\psi, \phi \in H^s((-\pi,\pi);\mathbb{C})$ we define the Hermitian bilinear form

$$\langle\!\langle \phi, \psi \rangle\!\rangle := \frac{1}{N} \sum_{\nu=0}^{N-1} \phi(\theta_\nu)\bar{\psi}(\theta_\nu)\,, \quad \theta_\nu := \frac{2\pi\nu}{N} - \pi\,, \tag{1.17}$$

which is called *the discrete inner product of* ψ *and* ϕ. The associate norm is defined by

$$|\!|\!|\phi|\!|\!|^2 := \langle\!\langle \phi, \phi \rangle\!\rangle := \frac{1}{N} \sum_{\nu=0}^{N-1} |\phi(\theta_\nu)|^2\,. \tag{1.18}$$

∎

Remark 1.28. Notice that by the last definition the approximating Fourier coefficients of f (see Definition 1.24) can be written as

$$\hat{\alpha}_k = \langle\!\langle f, \exp_k \rangle\!\rangle\,.$$

∎

Lemma 1.29. *Let* $N \in \mathbb{N}$ *and* $s > 1/2$*. For* $j, k \in \mathbb{Z}$ *we have*

$$\langle\!\langle \exp_j, \exp_k \rangle\!\rangle = \begin{cases} (-1)^{j-k} & j-k = lN \text{ for } l \in \mathbb{Z}\,, \\ 0 & \text{else}\,. \end{cases} \tag{1.19}$$

In particular, on the space

$$\hat{P}_{T[N]} := \mathtt{span}\{\exp_k : -N/2 < k \le N/2\} \tag{1.20}$$

the functions

$$\{\exp_k : -N/2 < k \le N/2\} \tag{1.21}$$

are an orthonormal system with respect to the discrete inner product defined in (1.17).

Proof. Let $j, k \in \mathbb{Z}$. Then

$$\begin{aligned}\langle\!\langle \exp_j, \exp_k \rangle\!\rangle =& \frac{1}{N} \sum_{\nu=0}^{N-1} \exp(\mathtt{i} j\theta_\nu) \exp(-\mathtt{i} k\theta_\nu) \\ =& \frac{1}{N} \exp\big(-\mathtt{i}\pi(j-k)\big) \sum_{\nu=0}^{N-1} \exp\big(2\pi\mathtt{i}(j-k)\nu/N\big) \\ =& \frac{(-1)^{j-k}}{N} \sum_{\nu=0}^{N-1} \Big(\exp\big(2\pi\mathtt{i}(j-k)/N\big)\Big)^\nu .\end{aligned}$$

Thus,

$$\langle\!\langle \exp_j, \exp_k \rangle\!\rangle = \begin{cases} \dfrac{(-1)^{j-k}}{N} \displaystyle\sum_{\nu=0}^{N-1} \big(\exp(2\pi\mathtt{i}l)\big)^\nu = (-1)^{j-k}, & j-k = lN,\, l \in \mathbb{Z}, \\ \dfrac{(-1)^{j-k}}{N} \dfrac{1-\exp\big(2\pi\mathtt{i}(j-k)\big)}{1-\exp\big(2\pi\mathtt{i}(j-k)/N\big)} = 0, & \text{else.} \end{cases}$$

□

We emphasize that the orthonormal system (1.21) defined in Lemma 1.29 has exactly N elements, independent whether N is even or odd.

Theorem 1.30. *Let $n < N/2$ and $\hat{p}_n$ as in* (1.16). *Let $f \in H^s((-\pi, \pi); \mathbb{C})$ with $s > 1/2$. Then for every trigonometric polynomial $p_n \in P_{T[n]} \backslash \{\hat{p}_n\}$ of degree n we have*

$$\sum_{\nu=0}^{N-1} |\hat{p}_n(\theta_\nu) - f(\theta_\nu)|^2 < \sum_{\nu=0}^{N-1} |p_n(\theta_\nu) - f(\theta_\nu)|^2 .$$

Proof. Assume $n < N/2$. Then the functions $\exp(\mathtt{i}k\theta)$, $-n \le k \le n$ are an orthonormal basis of $P_{T[n]}$ with respect to the discrete inner product $\langle\!\langle\, , \rangle\!\rangle$. By Theorem 1.9 and the identity $\hat{\alpha}_k = \langle\!\langle \exp_k, f \rangle\!\rangle$ the best approximation of f on $P_{T[n]}$, with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, is given by

$$\sum_{k=-n}^{n} \langle\!\langle \exp_k, f \rangle\!\rangle \exp_k = \sum_{k=-n}^{n} \hat{\alpha}_k \exp_k = \hat{p}_n .$$

Using that

$$\|\!|f - \hat{p}_n\|\!|^2 = \langle\!\langle \hat{p}_n - f, \hat{p}_n - f \rangle\!\rangle = \frac{1}{N} \sum_{\nu=0}^{N-1} |\hat{p}_n(\theta_\nu) - f(\theta_\nu)|^2$$

the assertion follows. □

Remark 1.31. Notice that Theorem 1.30 states that the trigonometric interpolation polynomial of order n defined in (1.16) provides the best approximation of f with respect to the discrete inner product $\langle\!\langle\,,\rangle\!\rangle$. In general, the interpolation error $\sum_{\nu=0}^{N-1} |\hat{p}_n(\theta_\nu) - f(\theta_\nu)|^2$ does not vanish. However, if we choose

$$n = n_{\max} := \lfloor N/2 \rfloor \tag{1.22}$$

then the trigonometric interpolation polynomial $\hat{p}_n$ interpolates the function values $f(\theta_\nu)$, $\nu = 0, 1, \dots, N-1$, and, as a consequence, the interpolation error vanishes. ■

Theorem 1.32. *Let $N \in \mathbb{N}$ and n_{max} be maximal as defined in* (1.22)*. Then the trigonometric interpolation polynomial $\hat{p}_{n_{max}}$ defined in* (1.16) *interpolates f at all knots θ_ν, $\nu = 0, \dots, N-1$, i.e., $\|\!|f - \hat{p}_{n_{max}}\|\!| = 0$.*

Proof. According to Lemma 1.29 the functions $\exp_k$, $-N/2 < k \le N/2$ are an orthonormal basis on $\hat{P}_{T[N]}$ with respect to the discrete inner product $\langle\!\langle\,,\rangle\!\rangle$. Thus, the vectors $\mathbf{v}_k = [\exp_k(\theta_\nu)]_{\nu=0}^{N-1} \in \mathbb{C}^N$, $-N/2 < k \le N/2$ are an orthonormal basis of $\mathbb{C}^N$. Consequently, the vector $[f(\theta_\nu)]_{\nu=0}^{N-1}$ is represented by a unique linear combination of the basis vectors $\mathbf{v}_k$. We define the index set $I := \{k \in \mathbb{N} : -N/2 < k \le N/2\}$ and obtain

$$[f(\theta_\nu)]_{\nu=0}^{N-1} = \sum_{k \in I} \tilde{c}_k \mathbf{v}_k\,, \quad \tilde{c}_k \in \mathbb{C}\,.$$

Hence, the trigonometric polynomial

$$\tilde{p}(\theta) := \sum_{k \in I} \tilde{c}_k \exp(\mathtt{i} k \theta)$$

belonging to the space $\hat{P}_{T[N]}$ satisfies

$$\langle\!\langle \tilde{p} - f, \tilde{p} - f \rangle\!\rangle = 0\,,$$

which shows that $\|\tilde{p} - f\| = 0$, and as a consequence (cf. Theorem 1.9), $\tilde{p}$ is the best approximation of f on $\hat{P}_{T[N]}$. One the other hand we know that $\hat{p}_{n_{\max}}$ is a best approximation of f on $\hat{P}_{T[N]}$, with respect to the discrete inner product, too (cf. Theorem 1.30 and a similar proof in case of $N = 2n_{\max}$). Thus $\tilde{p} = \hat{p}_{n_{\max}}$ and $\|f - \hat{p}_{n_{\max}}\| = 0$. □

The following Lemma is useful for deriving error estimates for trigonometric interpolation polynomials. It gives also a remarkable insight in the relation between Fourier coefficients and approximating Fourier coefficients of a function.

Lemma 1.33. *Let $f \in H^s((-\pi, \pi); \mathbb{C})$, $s > 1/2$ with approximating Fourier coefficients $\hat{\alpha}_k$ as defined in* (1.15)*. Then*

$$\hat{\alpha}_k = \sum_{l=-\infty}^{\infty} (-1)^{lN} \alpha_{k+lN} \,.$$

Proof. According to Remark 1.23 the Fourier series of f is uniformly convergent. Then from Lemma 1.29 it follows that

$$\begin{aligned}
\hat{\alpha}_k &= \frac{1}{N} \sum_{j=0}^{N-1} \underbrace{\left(\sum_{\nu=-\infty}^{\infty} \alpha_\nu \exp(\mathtt{i}\nu\theta_j) \right)}_{\text{formal Fourier series of } f} \exp(-\mathtt{i}k\theta_j) \\
&= \sum_{\nu=-\infty}^{\infty} \alpha_\nu \langle\!\langle \exp_\nu, \exp_k \rangle\!\rangle \\
&= \sum_{\substack{\nu \in \mathbb{Z} \\ \nu - k = lN}} (-1)^{\nu-k} \alpha_\nu \\
&= \sum_{l=-\infty}^{\infty} (-1)^{lN} \alpha_{k+lN} \,.
\end{aligned}$$

□

So far we have derived L^2 estimates for the Fourier series, and showed best approximation properties with respect to a discrete Euclidean inner product. Now, we consider L^∞ estimates which show a different behavior.

Theorem 1.34. *Let $f \in H^s((-\pi,\pi);\mathbb{C})$, $s > 1/2$, and denote by $\hat{p}_n$ the trigonometric interpolation polynomial of f as defined in* (1.16). *Then*

$$\|f - \hat{p}_n\|_{L^\infty((-\pi,\pi);\mathbb{C})} \le 2\left(\frac{2s}{\pi(2s-1)}\right)^{1/2} n^{1/2-s} \|f\|_{H^s((-\pi,\pi);\mathbb{C})} .$$

Proof. For convenience we define the index set

$$I := \{k \in \mathbb{N} : -N/2 < k < 1-n \text{ or } n < k \le N/2\} .$$

The uniform convergence of the Fourier series (cf. Remark 1.23) implies that $f(\theta) = \sum_{j=-\infty}^{\infty} \alpha_j \exp(\mathtt{i} j\theta)$, which can be expanded to

$$\sum_{l\in\mathbb{Z}} \sum_{-N/2<k\le N/2} \alpha_{k+lN} \exp(\mathtt{i}(k+lN)\theta) = \sum_{l\in\mathbb{Z}} \sum_{k=1-n}^{n} \alpha_{k+lN} \exp\big(\mathtt{i}(k+lN)\theta\big) + \sum_{l\in\mathbb{Z}} \sum_{k\in I} \alpha_{k+lN} \exp\big(\mathtt{i}(k+lN)\theta\big) .$$

Then, from the definition of the trigonometric interpolation polynomial $\hat{p}_n$ (see Definition 1.25) and Lemma 1.33, it follows that

$$f(\theta) - \hat{p}_n(\theta) = \sum_{k=1-n}^{n} \sum_{l\in\mathbb{Z}} \alpha_{k+lN} \Big(\exp\big(\mathtt{i}(k+lN)\theta\big) - (-1)^{lN} \exp(\mathtt{i}k\theta)\Big) + \sum_{k\in I} \sum_{l\in\mathbb{Z}} \alpha_{k+lN} \Big(\exp\big(\mathtt{i}(k+lN)\theta\big)\Big) .$$

For $1-n \le k \le n$ we have

$$\Big|\exp\big(\mathtt{i}(k+lN)\theta\big) - (-1)^{lN} \exp(\mathtt{i}k\theta)\Big| = 0 \text{ if } l = 0 ,$$
$$\Big|\exp\big(\mathtt{i}(k+lN)\theta\big) - (-1)^{lN} \exp(\mathtt{i}k\theta)\Big| \le 2 \text{ else} .$$

This shows that

$$\begin{aligned} |f(\theta) - \hat{p}_n(\theta)| &\le \sum_{k=1-n}^{n} \sum_{\substack{l\in\mathbb{Z}\\ l\ne 0}} 2\,|\alpha_{k+lN}| + \sum_{k\in I} \sum_{l\in\mathbb{Z}} |\alpha_{k+lN}| \\ &\le \sum_{k=1-n}^{n} \sum_{\substack{l\in\mathbb{Z}\\ l\ne 0}} 2\,|\alpha_{k+lN}| + \sum_{k\in I} \sum_{l\in\mathbb{Z}} 2\,|\alpha_{k+lN}| \\ &= 2\left(\sum_{\nu=-\infty}^{-n} |\alpha_\nu| + \sum_{\nu=n+1}^{\infty} |\alpha_\nu|\right) \end{aligned}$$

$$\leq 2 \sum_{|\nu|\geq n} |\alpha_\nu|$$
$$= 2 \sum_{|\nu|\geq n} |\nu|^{-s} \left(|\nu|^s\right) |\alpha_\nu| \,.$$

Then, from the Cauchy–Schwarz inequality it follows that

$$\begin{aligned}
|f(\theta) - \hat{p}_n(\theta)|^2 &\leq 4 \sum_{|\nu|\geq n} |\nu|^{-2s} \sum_{|\nu|\geq n} |\nu|^{2s} |\alpha_\nu|^2 \\
&\leq 8 \left(n^{-2s} + \int_n^\infty t^{-2s} ds \right) \sum_{|\nu|\geq n} |\nu|^{2s} |\alpha_\nu|^2 \\
&\leq \frac{4}{\pi} \left(n^{-2s} + \frac{n^{1-2s}}{2s-1} \right) \|f\|^2_{H^s((-\pi,\pi);\mathbb{C})} \\
&\leq \frac{4}{\pi} \frac{2s}{2s-1} n^{1-2s} \|f\|^2_{H^s((-\pi,\pi);\mathbb{C})} \,.
\end{aligned}$$

□

Remark 1.35. The estimate of Theorem 1.34 is order optimal. The Sobolev space $H^s((-\pi,\pi);\mathbb{C})$ with $s < 1/2$ contains discontinuous functions. Consequently, an element of this space cannot be uniformly approximated by trigonometric polynomials (since they are continuous). This is reflected in Theorem 1.34 above showing that the error estimates become worse for $s \to 1/2$. Better error estimates can be derived using other norms: For a 2π–periodic function $f \in C^s(\mathbb{R}), s \in \mathbb{N}$, one can show that

$$\|f - \hat{p}_n\|_{L^\infty((-\pi,\pi);\mathbb{R})} \leq C n^{-s} \log(n) \left\| f^{(s)} \right\|_{L^\infty((-\pi,\pi);\mathbb{R})} \,.$$

For a proof of this result we refer to [6]. ■

The following result shows that this approximation is optimal in the following sense:

Theorem 1.36. *Let $s > 1/2$. For every function $g \in H^s((-\pi,\pi);\mathbb{C})$*

$$\left| \int_{-\pi}^{\pi} g(\theta) d\theta - \frac{2\pi}{N} \sum_{\nu=0}^{N-1} g(\theta_\nu) \right| \leq c_s \|g\|_{H^s((-\pi,\pi);\mathbb{C})} h^s \,,$$

where $h = 2\pi/N$, θ_ν as defined in (1.14) *and c_s is a positive constant only depending on s.*

Proof. Let $\hat{\alpha}_k$ be the approximating Fourier coefficients and α_k be the Fourier coefficients of the function $g \in H^s((-\pi,\pi);\mathbb{C})$, $s > 1/2$. Since $s > 1/2$ the Fourier series of g is uniformly convergent, i.e., $g = \sum_{k=-\infty}^{\infty} \alpha_k \exp_k$. Utilizing Definition 1.24 of the approximating Fourier coefficients and Lemma 1.33 shows that

$$\frac{1}{N}\sum_{\nu=0}^{N-1} g(\theta_\nu) = \hat{\alpha}_0 = \alpha_0 + \sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} (-1)^{lN}\alpha_{lN} \ .$$

From the definition of the Fourier coefficients it follows that for $k = 0$ we obtain the mean value of the corresponding function, i.e.,

$$2\pi\alpha_0 = \int_{-\pi}^{\pi} g(\theta)\exp(\mathtt{i}0\theta)\, d\theta = \int_{-\pi}^{\pi} g(\theta)d\theta \ .$$

Therefore, by using the Cauchy–Schwarz inequality for infinite sums and the definition of the norm on the space $H^s((-\pi,\pi);\mathbb{C})$ (see Theorem 1.22), we obtain

$$\begin{aligned}
\left|\int_{-\pi}^{\pi} g(\theta)d\theta - \frac{2\pi}{N}\sum_{j=0}^{N-1} g(\theta_j)\right| &\le 2\pi \sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} |\alpha_{lN}| = 2\pi \sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} \left|(lN)^{-s}(lN)^s\alpha_{lN}\right| \\
&\le 2\pi \left(\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} |l|^{-2s} N^{-2s}\right)^{1/2} \left(\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} |lN|^{2s} |\alpha_{lN}|^2\right)^{1/2} \\
&\le \frac{2\pi}{N^s}\left(\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} |l|^{-2s}\right)^{1/2} \left(\sum_{l\in\mathbb{Z}} (|lN|+1)^{2s} |\alpha_{lN}|^2\right)^{1/2} \\
&= \frac{2\pi}{N^s}\left(2\sum_{l=1}^{\infty} l^{-2s}\right)^{1/2} \frac{1}{\sqrt{2\pi}} \|g\|_{H^s((-\pi,\pi);\mathbb{C})} \\
&= \frac{2\sqrt{\pi}}{N^s}\left(\sum_{l=1}^{\infty} l^{-2s}\right)^{1/2} \|g\|_{H^s((-\pi,\pi);\mathbb{C})} \ .
\end{aligned}$$

For $s > 1/2$ the series $\sum_{l=1}^{\infty} l^{-2s}$ is convergent, and thus the assertion follows. □

1.4 Gibbs–Phenomena

Errors which occure when one approximates a discontinuous function by its Fourier series expansion are called *Gibbs-phenomena*. Exemplarily we consider the approximation of the Heavyside function defined as

$$f(t) := \begin{cases} -1, & -\pi < t < 0\,, \\ +1, & 0 \le t < \pi\,. \end{cases}$$

Since f is an odd function, its Fourier series expansion only consists of sine terms and is given by

$$f(t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\bigl((2n-1)t\bigr)\,.$$

We define the N–th partial sum f_N by

$$f_N(t) := \frac{4}{\pi} \sum_{n=1}^{N} \frac{1}{2n-1} \sin\bigl((2n-1)t\bigr)\,.$$

Since

$$\frac{\sin\bigl((2n-1)t\bigr)}{2n-1} = \int_0^t \cos\bigl((2n-1)\tau\bigr)\,d\tau\,,$$

the N–th partial sum can be rewritten as

$$f_N(t) = \frac{4}{\pi} \int_0^t \sum_{n=1}^{N} \cos\bigl((2n-1)\tau\bigr)\,d\tau\,.$$

Applying the addition theorem for the sine function and reducing the arising telescoping sum we obtain

$$\begin{aligned} 2\sin(\tau) \sum_{n=1}^{N} \cos\bigl((2n-1)\tau\bigr) &= \sum_{n=1}^{N} \Bigl(\sin\bigl((2n-1+1)\tau\bigr) + \sin\bigl((1-2n+1)\tau\bigr) \Bigr) \\ &= \sum_{n=1}^{N} \sin(2n\tau) - \sin\bigl((2n-2)\tau\bigr) \\ &= \sin(2N\tau), \end{aligned}$$

and, as a consequence, we end up with

$$\sum_{n=1}^{N} \cos\big((2n-1)\tau\big) = \frac{\sin(2N\tau)}{2\sin(\tau)}\,.$$

Thus, f_N can be rewritten as

$$f_N(t) = \frac{2}{\pi}\int_0^t \frac{\sin(2N\tau)}{\sin(\tau)}\,d\tau\,.$$

Since

$$f_N'(t) = \frac{2}{\pi}\frac{\sin(2Nt)}{\sin(t)},$$

the extrema of f_N are located at $t_k := \pi k/(2N)$. Ordering the maxima consecutively the first maximum is at $t_1 = \pi/(2N)$ and

$$f_N\Big(\frac{\pi}{2N}\Big) = \frac{2}{\pi}\int_0^{\pi/2N} \frac{\sin(2N\tau)}{\sin(\tau)}\,d\tau\,.$$

For N sufficiently large we have in $[0, \pi/2N]$

$$\sin(\tau) \approx \tau$$

and, consequently,

$$f_N\Big(\frac{\pi}{2N}\Big) \approx \frac{2}{\pi}\int_0^{\pi/2N} \frac{\sin(2N\tau)}{\tau}\,d\tau = \frac{2}{\pi}\int_0^{\pi} \frac{\sin(x)}{x}\,dx \approx 1.17898\,.$$

Thus, the first maximum of the approximation f_N exceeds the actual function value $+1$ of f at the point $\pi/(2N)$. Relative to the height of the discontinuity of f the overshoot of the approximation is about 9%.

Gibbs–phenomena are well investigated. For instance, one can show that the relative overshoot of the approximation by finite Fourier sums with respect to the height of the discontinuity is *always* about 9%, independent of the order N of the approximation f_N. In addition, one can prove that the first extremum of the approximation before and after the discontinuity is indeed an overshoot or an undershoot, respectively (depending on whether the function value increases or decreases by the discontinuity).

Gibbs–phenomena also perfectly show that, in general, the Fourier series expansion approximates a function f best in the L^2–norm but not necessarily in the L^∞–norm.

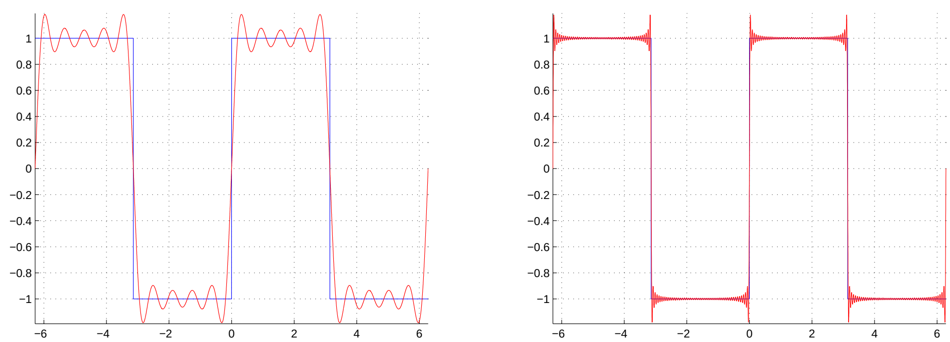

Figure 1.1: Gibbs-Phenomenon: Approximation of the 2π-periodic extended Heavyside function (blue) by its Fourier series expansion up to order N (red). Independent of the order of approximation Gibbs-Phenomenon occurs due to the discontinuity of the underlying function. The relative overshoot and undershoot, respectively, is about 9% of the height of the discontinuity. Left: Approximation of order $N = 5$. Right: Approximation of order $N = 50$.

1.5 Fast Fourier Transform

In the following we assume that $2n = N$, $n \in \mathbb{N}$. The mapping which maps the values $y_j = f(\theta_j)$, $j = 0, \ldots, N-1$ of a periodic function f at the discrete knots $\theta_j = 2\pi j/N$, $j = 0, \ldots, N-1$, to its approximating Fourier coefficients $\hat{\alpha}_k$, $1-n \le k \le n$, is called *discrete Fourier transform* (DFT). The discrete Fourier transform is used to find the best trigonometric interpolation polynomial, which is a smooth approximation of the function f.

Setting $\omega = \exp(-2\pi i/N)$, the discrete Fourier transform can be realized by the following matrix–vector multiplication

$$\begin{bmatrix} c_0 \\ \vdots \\ c_n \\ c_{n+1} \\ \vdots \\ c_{N-1} \end{bmatrix} := N \begin{bmatrix} \hat{\alpha}_0 \\ \vdots \\ \hat{\alpha}_n \\ \hat{\alpha}_{1-n} \\ \vdots \\ \hat{\alpha}_{-1} \end{bmatrix} = \begin{bmatrix} \omega^0 & \omega^0 & \ldots & \omega^0 \\ \omega^0 & \omega^1 & \ldots & \omega^{N-1} \\ \omega^0 & \omega^2 & \ldots & \omega^{2(N-1)} \\ \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{N-1} & \ldots & \omega^{(N-1)^2} \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix},$$

which we write in abbreviated form

$$c = Fy \, . \tag{1.23}$$

The symmetric, complex valued matrix F is called *Fourier matrix*. From Lemma 1.29 it follows that

$$F^*F = N \cdot I\,,$$

where F^* is the adjoint of F. That is, $F/\sqrt{N}$ is a unitary matrix which satisfies

$$\frac{1}{\sqrt{N}}F^* = \Big(\frac{F}{\sqrt{N}}\Big)^{-1} = \sqrt{N}F^{-1}\,. \tag{1.24}$$

Using (1.23) the approximating Fourier coefficients $\hat{\alpha}_k$ can be calculated for the best trigonometric interpolation polynomial. On the other hand, according to (1.24) the fast Fourier transform can be used to calculate $y_j = \hat{p}_n(\theta_j)$ of a trigonometric interpolation polynomial with coefficients $\hat{\alpha}_k$. To see that discrete Fourier transform is basically the same as the inverse discrete Fourier transform, we first notice that F is symmetric and by (1.23) we see that

$$y = F^{-1}c = \frac{1}{N}F^*c = \frac{1}{N}\overline{F\bar{c}}\,.$$

This formula is the basis of the *inverse discrete Fourier transform* (IDFT).

Both systems (1.23) and (1.5) are solvable by N^2 multiplications. However, there exist algorithms which only require $\mathcal{O}\big(N\log(N)\big)$ operations. Such algorithms are called *fast Fourier transform* (FFT) and *inverse fast Fourier transform* (IFFT), respectively.

To derive the FFT we assume that $N = 2^p$, $p \in \mathbb{N}$ and set $n = N/2$. The basis of the FFT is the following lemma.

Lemma 1.37. *Let $M = 2m$ and define*

$$\gamma_j := \sum_{\nu=0}^{M-1} \eta_\nu\, \omega_M^{j\nu}\,, \quad j = 0,\dots,M-1\,,$$

with $\omega_M := \exp(-\mathtt{i}2\pi/M)$. Setting $\omega_m = \omega_M^2$ we have for $l = 0,\dots,m-1$

$$\gamma_{2l} = \sum_{\nu=0}^{m-1} \eta_\nu^{(+)}\omega_m^{l\nu}\,, \qquad \eta_\nu^{(+)} = \eta_\nu + \eta_{\nu+m}\,,$$

$$\gamma_{2l+1} = \sum_{\nu=0}^{m-1} \eta_\nu^{(-)}\omega_m^{l\nu}\,, \qquad \eta_\nu^{(-)} = (\eta_\nu - \eta_{\nu+m})\,\omega_M^{\nu}\,.$$

Proof. For even indices we have

$$\begin{aligned}
\gamma_{2l} &= \sum_{\nu=0}^{M-1} \eta_\nu\, \omega_M^{2l\nu} = \sum_{\nu=0}^{m-1} \Big(\eta_\nu\, \omega_M^{2l\nu} + \eta_{\nu+m}\, \omega_M^{2l\nu+2lm}\Big) \\
&= \sum_{\nu=0}^{m-1} \Big(\eta_\nu\, \omega_M^{2l\nu} + \eta_{\nu+m}\, \omega_M^{2l\nu}\Big) \\
&= \sum_{\nu=0}^{m-1} (\eta_\nu + \eta_{\nu+m})\, \omega_m^{l\nu}\,.
\end{aligned}$$

For odd indices we have

$$\begin{aligned}
\gamma_{2l+1} &= \sum_{\nu=0}^{m-1} \Big(\eta_\nu\, \omega_M^{(2l+1)\nu} + \eta_{\nu+m}\, \omega_M^{(\nu+m)(2l+1)}\Big) \\
&= \sum_{\nu=0}^{m-1} \Big(\eta_\nu + \eta_{\nu+m}\, \omega_M^{(2l+1)m}\Big) \omega_M^{(2l+1)\nu} \\
&= \sum_{\nu=0}^{m-1} (\eta_\nu - \eta_{\nu+m})\, \omega_M^{\nu}\, \omega_m^{l\nu}\,. \qquad \square
\end{aligned}$$

Since γ_{2l} and γ_{2l+1} have the same form as γ_l, but only half of the coefficients, the calculation can be performed recursively. In the following we summarize recursive algorithms for FFT and IFFT:

Algorithm 1: Recursive scheme of the discrete Fourier transform

function $\gamma = \texttt{dft}(\eta, \omega, M)$;
// Calculates the discrete Fourier transform of the given vector $\eta \in \mathbb{C}^M$ for given M (even) and $\omega := \exp(-\mathrm{i}2\pi/M)$.
// CAUTION: $\cdot$ indicates componentwise multiplication of two vectors!
$\eta^{(+)} = \eta(0 : M/2-1) + \eta(M/2 : M-1)$;
$\eta^{(-)} = \big(\eta(0 : M/2-1) - \eta(M/2 : M-1)\big) \cdot [1, \omega, \ldots, \omega^{M/2-1}]$;
if $M = 2$ **then**
  $\gamma = [\eta^{(+)}, \eta^{(-)}]$;
else
  $\gamma(0:2:M-2) = \texttt{dft}(\eta^{(+)}, \omega^2, M/2)$;
  $\gamma(1:2:M-1) = \texttt{dft}(\eta^{(-)}, \omega^2, M/2)$;
end

Algorithm 2: Fast Fourier transform

```
function c = fft(y, N);
// Calculates the fast Fourier transform of the given
   vector y ∈ ℂ^N for given N (even!).
ω = exp(−i2π/N);
c = dft(y, ω, N);
```

Algorithm 3: Inverse fast Fourier transform

```
function y = ifft(c, N);
// Calculates the inverse fast Fourier transform of the
   given vector c ∈ ℂ^N for given N (even!).
y = conj(fft(conj(c), N))/N;
```

Numerical complexity of FFT: Let $N = 2^p$, i.e., $p = \log_2(N)$ and assume the values $\omega^0, \dots, \omega^{N-1}$ to be precalculated. Then in every recursive step N complex adds and $N/2$ complex multiplications have to be performed. In total there are p recursive steps required. Thus, the total complexity of the FFT is given by

$$N \log_2(N) \text{ complex adds}$$

and

$$\frac{N}{2} \log_2(N) \text{ complex multiplications.}$$

Example 1.38. In Figure 1.2 the approximate Fourier coefficients $\hat{\alpha}_k$ of the real valued function $f\colon (-\pi, \pi) \to \mathbb{R}$, $f(x) := (x-\pi)^2(x+\pi)^2$ are shown. Since f is an even function we know from theory that the imaginary part of $\hat{\alpha}_k$ should vanish. Indeed, the values of the imaginary part shown in Figure 1.2 are in the range of the machine accuracy, and can be seen equal to zero. The computation was performed with the Matlab built-in functions `fft` and `fftshift`. ■

The *discrete cosine transform* (DCT) provides a transformation between real functions. The results are comparable by using the FFT on an evenly extended vector around zero.

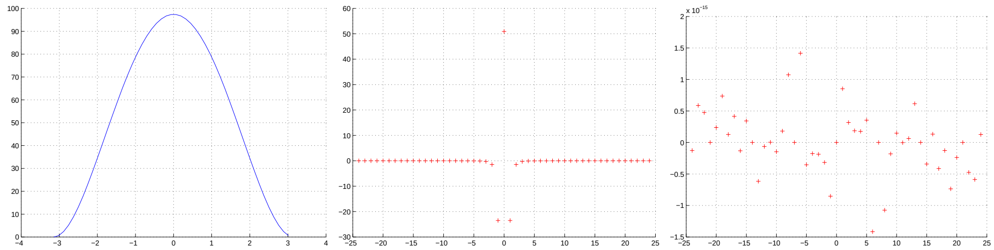

Figure 1.2: Fast Fourier transform (FFT) of the even function $f(x) := (x-\pi)^2(x+\pi)^2$ on $(-\pi,\pi)$. Left: Signal in the time domain. Center: Real part of the approximating Fourier coefficients $\hat{\alpha}_k$. Right: Imaginary part of the approximating Fourier coefficients $\hat{\alpha}_k$. Computation was perfomed with the Matlab built-in functions `fft` and `fftshift`. The function was uniformly sampled with stepsize equal to $2\pi/100$.

Definition 1.39. Assume $f(\nu) := f(\theta_\nu^C)$ a given signal sampled at the points $\theta_\nu^C := \frac{2\nu+1}{2N}\pi$, $0 \le \nu \le N-1$. For $k \in \mathbb{N}$ define

$$w_k := \begin{cases} 1/\sqrt{2} & \text{if } k = 0\,, \\ 1 & \text{else}\,. \end{cases}$$

The transform

$$\hat{\alpha}_k^C := \sqrt{\frac{2}{N}} \sum_{\nu=0}^{N-1} w_k f(\theta_\nu^C) \cos(k\theta_\nu^C)\,, \quad 0 \le k \le N-1\,, \tag{1.25}$$

is called *discrete cosine transform* (DCT) of the vector $f(\nu)$ and

$$f_\nu^C := \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} w_k\, \hat{\alpha}_k^C \cos(k\theta_\nu^C)\,, \quad 0 \le \nu \le N-1\,, \tag{1.26}$$

the corresponding *inverse discrete cosine transform* (IDCT) of the vector $\hat{\alpha}_k^C$. ■

Remark 1.40. Notice that in (1.25) the weights w_k are independent of the summation index, whereas in (1.26) the are dependent. Thus, in contrast to the DFT, the transforms of Definition 1.39 are not symmetric. ■

1.6 Fourier Integral Representation

Let $f \in L^2\big((-l,l);\mathbb{C}\big)$. Then, using the transformation

$$\begin{aligned}\zeta\colon (-l,l) &\to (-\pi,\pi)\\ t &\mapsto \zeta(t) := \frac{t\pi}{l}\,.\end{aligned}$$

the square integrable function $g \in L^2\big((-\pi,\pi);\mathbb{C}\big)$ defined as

$$g := f \circ \zeta^{-1}$$

can be expanded into a Fourier series

$$\begin{aligned}g(s) &= g^r(s) + \mathtt{i}g^c(s)\\ &= \frac{a_0^r}{2} + \sum_{k=1}^{\infty}\big(a_k^r\cos(ks) + b_k^r\sin(ks)\big) +\\ &\qquad + \mathtt{i}\left(\frac{a_0^c}{2} + \sum_{k=1}^{\infty}\big(a_k^c\cos(ks) + b_k^c\sin(ks)\big)\right)\end{aligned}$$

where the coefficients a_k^r and b_k^r denote the Fourier coefficients of the real part g^r, and the coefficients a_k^c and b_k^c denote the Fourier coefficients of the imaginary part g^c of g. Thus, the Fourier series of f is given by

$$\begin{aligned}f(t) &= g\big(\zeta(t)\big)\\ &= \frac{a_0^r}{2} + \sum_{k=1}^{\infty}\left(a_k^r\cos\left(\frac{k\pi t}{l}\right) + b_k^r\sin\left(\frac{k\pi t}{l}\right)\right)\\ &\qquad + \mathtt{i}\left(\frac{a_0^c}{2} + \sum_{k=1}^{\infty}\left(a_k^c\cos\left(\frac{k\pi t}{l}\right) + b_k^c\sin\left(\frac{k\pi t}{l}\right)\right)\right)\,.\end{aligned}$$

The Fourier coefficients a_k^r, $0 \le k \le \infty$ can be expressed as

$$\begin{aligned}a_k^r &= \frac{1}{\pi}\int_{-\pi}^{\pi} g^r(\xi)\cos(k\xi)\,d\xi = \frac{1}{\pi}\int_{-l}^{l} g^r\left(\frac{\pi\tau}{l}\right)\cos\left(\frac{k\pi\tau}{l}\right)\frac{\pi}{l}\,d\tau\\ &= \frac{1}{l}\int_{-l}^{l} f^r(\tau)\cos\left(\frac{k\pi\tau}{l}\right)d\tau\end{aligned}$$

where f^r denotes the real part of the function f. The other Fourier coefficients can be derived analogously. By defining

$$\begin{aligned} c_k &:= a_k^r + \mathtt{i} a_k^c = \frac{1}{l}\int_{-l}^{l} f(\tau)\cos\left(\frac{k\pi\tau}{l}\right) d\tau\,, \quad 0 \le k \le \infty\,, \\ d_k &:= b_k^r + \mathtt{i} b_k^c = \frac{1}{l}\int_{-l}^{l} f(\tau)\sin\left(\frac{k\pi\tau}{l}\right) d\tau\,, \quad 1 \le k \le \infty\,, \end{aligned} \tag{1.27}$$

the Fourier series of f can be written as

$$f(t) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\left(\frac{k\pi t}{l}\right) + d_k \sin\left(\frac{k\pi t}{l}\right). \tag{1.28}$$

Combining (1.27) with (1.28) and using the cosine summation theorem

$$\cos(t)\cos(\tau) + \sin(t)\sin(\tau) = \cos(t-\tau)$$

we obtain for $t \in (-l, l)$ the identity

$$f(t) = \frac{1}{2l}\int_{-l}^{l} f(\tau)\,d\tau + \sum_{k=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(\tau)\cos\left(\frac{k\pi}{l}(\tau - t)\right) d\tau. \tag{1.29}$$

In the following let $f \in L^1(\mathbb{R};\mathbb{C}) \cap L^2(\mathbb{R};\mathbb{C})$ and define $c := \|f\|_{L^1(\mathbb{R};\mathbb{C})}$. Then

$$\frac{1}{2l}\left|\int_{-l}^{l} f(\tau)\,d\tau\right| \le \frac{1}{2l}\int_{-l}^{l} |f(\tau)|\,d\tau \le \frac{c}{2l} \to 0 \qquad (l \to \infty)\,.$$

Setting

$$\omega_k := \frac{k\pi}{l}\,, \qquad 0 \le k \le \infty\,,$$

we find that

$$\Delta\omega_k := \omega_{k+1} - \omega_k = \frac{\pi}{l} \to 0 \quad (l \to \infty)\,.$$

Therefore,

$$\sum_{k=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(\tau)\cos\left(\frac{k\pi}{l}(\tau - t)\right) d\tau = \frac{1}{\pi}\sum_{k=1}^{\infty}\Delta\omega_{k-1}\int_{-l}^{l} f(\tau)\cos\big(\omega_k(\tau - t)\big)\,d\tau.$$

The term on the right hand side of the last identity is an approximation of the double integral

$$\frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(\tau)\cos\big(\omega(\tau-t)\big)\,d\tau\,d\omega\,.$$

A formal argument, by letting $l \to \infty$ in (1.29) shows that (this argument can be justified by methods from harmonic analysis) that for every function $f \in L^1(\mathbb{R};\mathbb{C}) \cap L^2(\mathbb{R};\mathbb{C})$

$$f(t) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(\tau)\cos\big(\omega(\tau-t)\big)\,d\tau\,d\omega\,. \tag{1.30}$$

This formula is called the *Fourier integral representation of f*.

Remark 1.41. The Fourier integral representation of f can be rewritten as

$$f(t) = \int_0^\infty \big(a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)\big)\,d\omega, \tag{1.31}$$

with

$$\begin{aligned} a(\omega) &:= \frac{1}{\pi}\int_{-\infty}^\infty f(\tau)\cos(\omega\tau)\,d\tau\,,\\ b(\omega) &:= \frac{1}{\pi}\int_{-\infty}^\infty f(\tau)\sin(\omega\tau)\,d\tau\,. \end{aligned} \tag{1.32}$$

■

Comparing (1.32) and (1.27) reveals the analogy between Fourier series expansion and Fourier integral representation. The discrete parameter k in (1.27) and (1.28) is replaced by the continuous parameter ω in (1.31) and (1.32), respectively.

1.7 Fourier Transform

Let $f \in L^1(\mathbb{R};\mathbb{C})$ be differentiable (to simplify the considerations). For fixed t and τ the function

$$\omega \to f(\tau)\cos\big(\omega(\tau-t)\big)$$

is even. Thus

$$\omega \to \int_{-\infty}^\infty f(\tau)\cos\big(\omega(\tau-t)\big)\,d\tau$$

is even too and, consequently, the Fourier integral representation can be rewritten as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) \cos\big(\omega(t-\tau)\big)\, d\tau\, d\omega. \tag{1.33}$$

We now turn our attention to analogous integral expressions involving the sine function. From $\big|\sin\big(\omega(\tau - t)\big)\big| \leq 1$ it follows that for every $f \in L^1(\mathbb{R};\mathbb{C})$

$$\omega \to \int_{-\infty}^{\infty} f(\tau) \sin\big(\omega(t-\tau)\big)\, d\tau$$

is well–defined and odd. Thus it follows that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) \sin\big(\omega(t-\tau)\big)\, d\tau\, d\omega$$
$$:= \lim_{A\to\infty} \int_{-A}^{A} \int_{-\infty}^{\infty} f(\tau) \sin\big(\omega(t-\tau)\big)\, d\tau\, d\omega = 0. \tag{1.34}$$

Both identities (1.33) and (1.34) together imply that

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) \exp\big(\mathtt{i}\omega(t-\tau)\big)\, d\tau\, d\omega\,, \tag{1.35}$$

where the outer integral has to be understood in the sense of a Cauchy principal value.

Formally, the right hand side of (1.35) can be decomposed into two operators, which are called Fourier transform and inverse Fourier transform, respectively.

Definition 1.42. We define the two operators

$$(\mathcal{F}f)(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) \exp(-\mathtt{i}\omega\tau) d\tau \tag{1.36}$$

and

$$(\mathcal{F}^{-1}\tilde{f})(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) \exp(\mathtt{i}\omega t) d\omega\ , \tag{1.37}$$

whenever the right hand side exists. Then $\mathcal{F}f$ is called the *Fourier transform of* f and $\mathcal{F}^{-1}\tilde{f}$ the *inverse Fourier transform of* $\tilde{f}$. ∎

Remark 1.43. The definition of the operators $\mathcal{F}$ and $\mathcal{F}^{-1}$ is far from being treated in a unified manner in literature. For instance, one finds that the Fourier transform is associated with the operator $\mathcal{F}^{-1}$ such that $\mathcal{F}$ is its inverse, and, in addition, often different multiplicative factors appear in the definitions.

Notice that the considerations for the derivation of the definitions above are up to a certain degree of heuristic character: Indeed, more advanced methods from Fourier analysis are necessary in order to prove that $\mathcal{F}$ is invertible with inverse $\mathcal{F}^{-1}$ on some function space (e.g. on $L^2(\mathbb{R};\mathbb{C})$). ■

1.8 Fourier Sine and Cosine Transform

In this section we introduce two transforms that are closely related to the Fourier transform.

Definition 1.44. We define the two operators

$$(\mathcal{F}_s f)(\omega) := \sqrt{\frac{2}{\pi}} \int_0^\infty f(\tau) \sin(\omega\tau)\, d\tau \tag{1.38}$$

and

$$(\mathcal{F}_c f)(\omega) := \sqrt{\frac{2}{\pi}} \int_0^\infty f(\tau) \cos(\omega\tau)\, d\tau\,, \tag{1.39}$$

whenever the right hand side exists. Then $\mathcal{F}_s f$ is called the *Fourier sine transform of f* and $\mathcal{F}_c f$ the *Fourier cosine transform of f*. ■

Some of the relations between the operators $\mathcal{F}$, $\mathcal{F}_c$, and $\mathcal{F}_s$ are listed below:

1. If f is an even function, then

$$\begin{aligned}(\mathcal{F} f)(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(t) \exp(-\mathtt{i}\omega t)\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(t) \cos(\omega t)\, dt \\ &= (\mathcal{F}_c f)(\omega)\,.\end{aligned}$$

2. If f is an odd function, then

$$\begin{aligned}(\mathcal{F}f)(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp(-\mathtt{i}\omega t)\, dt \\ &= \frac{-\mathtt{i}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \sin(\omega t)\, dt \\ &= -\mathtt{i}(\mathcal{F}_s f)(\omega)\,.\end{aligned}$$

3. In general every function f can be decomposed in the sum of its even part f_e and its odd part f_o, i.e.,

$$f_e(x) := \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) := \frac{f(x) - f(-x)}{2}\,.$$

Consequently, the Fourier transform can be decomposed into the sum of the Fourier cosine transform of the even part f_e and the Fourier sine transform of the odd part f_o:

$$\begin{aligned}\sqrt{2\pi}\,(\mathcal{F}f)(\omega) &= \int_{-\infty}^{\infty} f(\tau) \exp(-\mathtt{i}\omega\tau)\, d\tau = \int_{-\infty}^{\infty} (f_e + f_o)(\tau) \exp(-\mathtt{i}\omega\tau)\, d\tau \\ &= \bigg(\int_{-\infty}^{\infty} f_e(\tau) \cos(\omega\tau)\, d\tau - \mathtt{i} \int_{-\infty}^{\infty} f_e(\tau) \sin(\omega\tau)\, d\tau + \\ &\qquad \int_{-\infty}^{\infty} f_o(\tau) \cos(\omega\tau)\, d\tau - \mathtt{i} \int_{-\infty}^{\infty} f_o(\tau) \sin(\omega\tau)\, d\tau \bigg) \\ &= 2\left(\int_{0}^{\infty} f_e(\tau) \cos(\omega\tau)\, d\tau - \mathtt{i} \int_{0}^{\infty} f_o(\tau) \sin(\omega\tau)\, d\tau \right)\end{aligned}$$

Thus, we obtain

$$\mathcal{F}f = \mathcal{F}_c f_e - \mathtt{i}\mathcal{F}_s f_o\,.$$

1.9 Properties of the Fourier Transform

In this section we study some basic properties of the Fourier transform. First, we show some properties that are independent of the function space where the Fourier transform is defined on. Second, we study some theorems valid for the function space $L^1(\mathbb{R};\mathbb{C})$ and conclude the section with the investigation of the Fourier transform on the function space $L^2(\mathbb{R};\mathbb{C})$.

The following lemma can be shown independent of the underlying function space of the Fourier transform.

Lemma 1.45. *For $a \neq 0$ we define $f_a(t) := f(at)$ and $f_{[b]}(t) := f(t-b)$, $b \in \mathbb{R}$. Then*

$$(\mathcal{F}f_a)(\omega) = \frac{1}{|a|}(\mathcal{F}f)\Big(\frac{\omega}{a}\Big) \tag{1.40}$$

and

$$(\mathcal{F}f_{[b]})(\omega) = \exp(-\mathtt{i}\omega b)(\mathcal{F}f)(\omega)\ . \tag{1.41}$$

Proof. Let $a \neq 0$ and $b \in \mathbb{R}$. Using the definition of the Fourier transform we obtain

$$\begin{aligned}(\mathcal{F}f_a)(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(a\tau)\exp(-\mathtt{i}\omega\tau)\,d\tau \\ &= \frac{1}{|a|}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\exp\Big(-\mathtt{i}\frac{\omega}{a}y\Big)\,dy \\ &= \frac{1}{|a|}(\mathcal{F}f)\Big(\frac{\omega}{a}\Big)\,,\end{aligned}$$

and

$$\begin{aligned}(\mathcal{F}f_{[b]})(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau-b)\exp(-\mathtt{i}\omega\tau)\,d\tau \\ &= \exp(-\mathtt{i}\omega b)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\exp(-\mathtt{i}\omega y)\,d\tau \\ &= \exp(-\mathtt{i}\omega b)(\mathcal{F}f)(\omega)\ .\end{aligned}$$

□

Next, we focus on the function space $L^1(\mathbb{R};\mathbb{C})$.

Theorem 1.46. *Let $f \in L^1(\mathbb{R};\mathbb{C})$. Then $\mathcal{F}f$ is continuous and satisfies*

$$\lim_{|\omega|\to\infty}(\mathcal{F}f)(\omega) = 0\,.$$

Proof. Since $f \in L^1(\mathbb{R};\mathbb{C})$ we see that its Fourier transform exists for all $\omega \in \mathbb{R}$ as

$$\left|\int_{-\infty}^{\infty} f(\tau)\exp(-\mathtt{i}\omega\tau)\,d\tau\right| \leq \int_{-\infty}^{\infty}|f(\tau)|\,d\tau < \infty\,.$$

Let ω_1, $\omega_2 \in \mathbb{R}$ be arbitrary. From

$$(\mathcal{F}f)(\omega_1) - (\mathcal{F}f)(\omega_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)\big(\exp(-\mathtt{i}\omega_1\tau) - \exp(-\mathtt{i}\omega_2\tau)\big)\, d\tau$$

we obtain immediately that

$$|(\mathcal{F}f)(\omega_1) - (\mathcal{F}f)(\omega_2)| \leq \frac{\|f\|_{L^1(\mathbb{R};\mathbb{C})}}{\sqrt{2\pi}} \max_{\tau\in\mathbb{R}} |\exp(-\mathtt{i}\omega_1\tau) - \exp(-\mathtt{i}\omega_2\tau)|\ .$$

Since the complex valued exponential function is 2π-periodic it follows that

$$\max_{\tau\in\mathbb{R}} |\exp(-\mathtt{i}\omega_1\tau) - \exp(-\mathtt{i}\omega_2\tau)| = \max_{\tau\in[-\pi,\pi]} |\exp(-\mathtt{i}\omega_1\tau) - \exp(-\mathtt{i}\omega_2\tau)|\ .$$

For $\omega_1 \to \omega_2$ the function $\exp(-\mathtt{i}\omega_1\tau)$ converges uniformly to $\exp(-\mathtt{i}\omega_2\tau)$ for $\tau \in [-\pi, \pi]$. Thus the Fourier transform is continuous and the first part of the assertion is proven.

For the second part of the proof we first notice that for $\omega \neq 0$ we obtain

$$\begin{aligned}(\mathcal{F}f)(\omega) &= \int_{-\infty}^{\infty} f(\tau)\exp(-\mathtt{i}\omega\tau)\, d\tau = -\exp(-\mathtt{i}\pi)\int_{-\infty}^{\infty} f(\tau)\exp(-\mathtt{i}\omega\tau)\, d\tau \\ &= -\int_{-\infty}^{\infty} f(\tau)\exp\big(-\mathtt{i}\omega(\tau + \pi/\omega)\big)\, d\tau \\ &= -\int_{-\infty}^{\infty} f(\tau - \pi/\omega)\exp(-\mathtt{i}\omega\tau)\, d\tau\end{aligned}$$

and, as a consequence, we see that

$$\begin{aligned}2\,|(\mathcal{F}f)(\omega)| &= \left|\int_{-\infty}^{\infty} \big(f(\tau) - f(\tau - \pi/\omega)\big)\exp(-\mathtt{i}\omega\tau)\, d\tau\right| \\ &\leq \int_{-\infty}^{\infty} |f(\tau) - f(\tau - \pi/\omega)|\ d\tau\,.\end{aligned}$$

Since $f \in L^1(\mathbb{R};\mathbb{C})$ we infer from Lebesgue theory that

$$\lim_{|\omega|\to\infty} \int_{-\infty}^{\infty} |f(\tau) - f(\tau - \pi/\omega)|\ d\tau = 0\,,$$

which shows that

$$\lim_{|\omega|\to\infty} (\mathcal{F}f)(\omega) = 0\,.$$

□

Remark 1.47. Notice that the proof of Theorem 1.46 shows actually that the Fourier transform of an L^1-function is uniformly continuous. Indeed, using that for $\tau \in [-\pi, \pi]$

$$\left| \int_{\omega_2}^{\omega_1} -\mathtt{i}\tau \exp(-\mathtt{i}\omega\tau)\, d\omega \right| \leq |\tau|\, |\omega_1 - \omega_2| \leq \pi\, |\omega_1 - \omega_2|$$

uniform continuity follows immediately. ∎

Theorem 1.48. *Let $n \in \mathbb{N}_0$. Assume $f \in L^1(\mathbb{R}; \mathbb{C})$ and for $0 \leq k \leq n$*

$$g_k(t) := t^k f(t) \in L^1(\mathbb{R}; \mathbb{C})\,.$$

Then $(\mathcal{F}f)(\omega)$ is n-times differentiable and for $0 \leq k \leq n$ the k-th derivative of the Fourier transform of f equals

$$(\mathcal{F}f)^{(k)}(\omega) = \frac{(-\mathtt{i})^k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)\tau^k \exp(-\mathtt{i}\omega\tau)\, d\tau = (-\mathtt{i})^k (\mathcal{F}g_k)(\omega)\,.$$

Moreover, for $0 \leq k \leq n$ we have

$$\lim_{|\omega| \to \infty} (\mathcal{F}f)^{(k)}(\omega) = 0\,.$$

Proof. First we define $F(\omega, t) := f(t) \exp(-\mathtt{i}\omega t)$ and notice that $F(\cdot, t)$ is differentiable for almost every $t \in \mathbb{R}$. Thus, for $0 \leq k \leq n$ and almost every $t \in \mathbb{R}$

$$\left| \frac{\partial F^{(k)}}{\partial \omega^k}(\omega, t) \right| = \left| f(t) \exp(-\mathtt{i}\omega t)(-\mathtt{i}t)^k \right| = |t|^k\, |f(t)| =: g_k(t)\,.$$

By assumption g_k, $0 \leq k \leq n$ is integrable. Thus, we have shown that for every $0 \leq k \leq n$ the k-th derivative of F with respect to ω is uniformly dominated by the function g_k, i.e., for $0 \leq k \leq n$

$$\forall \omega \in \mathbb{R} : \left| \frac{\partial F^{(k)}}{\partial \omega^k}(\omega, t) \right| \leq g_k(t) \quad \text{for almost every } t \in \mathbb{R}\,.$$

As a consequence, integration and differentiation in

$$(\mathcal{F}f)^{(k)} = \frac{d^k}{d\omega^k}(\mathcal{F}f)(\omega) = \frac{1}{\sqrt{2\pi}} \frac{d^k}{d\omega^k} \int_{-\infty}^{\infty} f(\tau) \exp(-\mathtt{i}\omega\tau)\, d\tau$$

can be interchanged, which proves the assertion immediately. The second assertion follows from Theorem 1.46 using the fact that $t^k f(t) \in L^1(\mathbb{R}; \mathbb{C})$.□

The above theorems are based on the assumption $f \in L^1(\mathbb{R};\mathbb{C})$. In the following we study some results in the case of $f \in L^2(\mathbb{R};\mathbb{C})$.

Theorem 1.49 (Plancherel). *Let $f \in L^2(\mathbb{R};\mathbb{C})$. Then the Fourier transform of f as defined in* (1.36) *exists. In addition, the inverse Fourier transform (cf. Definition 1.42) exists for all functions $f \in L^2(\mathbb{R};\mathbb{C})$ too and*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mathcal{F}f)(\omega) \exp(\mathtt{i}\omega t)\, d\omega = f(t)\,,$$

that is, $(\mathcal{F}^{-1}\mathcal{F}f)(t) = f(t)$. Moreover,

$$\|\mathcal{F}f\|_{L^2(\mathbb{R};\mathbb{C})} = \|f\|_{L^2(\mathbb{R};\mathbb{C})} \tag{1.42}$$

which is called the isometric property of the Fourier transform on $L^2(\mathbb{R};\mathbb{C})$.

Proof. The proof of this theorem can be found in [3]. □

Theorem 1.50 (Convolution). *Let $k \in L^1(\mathbb{R};\mathbb{C})$ and $f \in L^2(\mathbb{R};\mathbb{C})$. The* convolution of k with f *is defined as*

$$(k * f)(t) := \int_{-\infty}^{\infty} k(t-\tau)\, f(\tau)\, d\tau\,.$$

and satisfies $k * f \in L^2(\mathbb{R};\mathbb{C})$ with

$$\|k * f\|_{L^2(\mathbb{R};\mathbb{C})} \le \|k\|_{L^1(\mathbb{R};\mathbb{C})}\, \|f\|_{L^2(\mathbb{R};\mathbb{C})}\,.$$

Moreover,

$$\big(\mathcal{F}(k * f)\big)(\omega) = \sqrt{2\pi}(\mathcal{F}k)(\omega)(\mathcal{F}f)(\omega)\,.$$

Proof. We have

$$\begin{aligned}
\big(\mathcal{F}(k * f)\big)(\omega) &= \left(\mathcal{F}\left(\int_{-\infty}^{\infty} k(t-\tau) f(\tau)\, d\tau\right)\right)(\omega) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(t-\tau) f(\tau)\, d\tau \exp(-\mathtt{i}\omega t)\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(y) \exp(-\mathtt{i}\omega y) f(\tau) \exp(-\mathtt{i}\omega\tau)\, d\tau\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(y) \exp(-\mathtt{i}\omega y)\, dy \int_{-\infty}^{\infty} f(\tau) \exp(-\mathtt{i}\omega\tau)\, d\tau \\
&= \sqrt{2\pi}(\mathcal{F}k)(\omega)(\mathcal{F}f)(\omega)\,.
\end{aligned}$$

Since

$$|(\mathcal{F}k)(\omega)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |k(t) \exp(-\mathtt{i}\omega t)| \; dt \leq \frac{1}{\sqrt{2\pi}} \|k\|_{L^1(\mathbb{R};\mathbb{C})}$$

it follows from the isometry property of the Fourier-Transform on $L^2(\mathbb{R};\mathbb{C})$ (see (1.42) that

$$\begin{aligned} \|k * f\|_{L^2(\mathbb{R};\mathbb{C})} &= \|\mathcal{F}(k * f)\|_{L^2(\mathbb{R};\mathbb{C})} \\ &= \sqrt{2\pi} \|(\mathcal{F}k)(\mathcal{F}f)\|_{L^2(\mathbb{R};\mathbb{C})} \\ &\leq \sqrt{2\pi} \|\mathcal{F}k\|_{L^\infty(\mathbb{R};\mathbb{C})} \|\mathcal{F}f\|_{L^2(\mathbb{R};\mathbb{C})} \\ &\leq \|k\|_{L^1(\mathbb{R};\mathbb{C})} \|f\|_{L^2(\mathbb{R};\mathbb{C})} . \end{aligned}$$

Notice that we also used that $\mathcal{F}k$ is continuous with $(\mathcal{F}k)(\omega) \to 0$ as $|\omega| \to \infty$ (see Theorem 1.46) and, as a consequence, $\|\mathcal{F}k\|_{L^\infty(\mathbb{R};\mathbb{C})}$ exists. □

Chapter 2

Sampling and Aliasing

In this chapter we focus on a well known theorem in the area of signal processing: the Shannon sampling theorem. It states that under certain conditions it is possible to reconstruct a band limited function from discrete sampling points. To that end, we first introduce the family of functions of interest.

Definition 2.1. A function $f \in L^2(\mathbb{R};\mathbb{C})$ is called *b–band limited*, $b > 0$, if for $\omega \notin [-b, b]$

$$(\mathcal{F}f)(\omega) = 0\,. \qquad \blacksquare$$

Theorem 2.2 (Shannon Sampling Theorem). *Let $f \in L^2(\mathbb{R};\mathbb{C})$ be b–band limited, $b > 0$. For $h \le \frac{\pi}{b}$ the function f is uniquely determined from the samples $f(hk)$, $k \in \mathbb{Z}$. More precisely,*

$$f(t) = \sum_{k\in\mathbb{Z}} f(hk)\,\mathrm{sinc}\Big(\frac{\pi}{h}(hk-t)\Big) \ \text{in } L^2(\mathbb{R};\mathbb{C})\,. \tag{2.1}$$

In addition, the Fourier transform of f is given by

$$(\mathcal{F}f)(\omega) = \frac{h}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} f(hk)\exp(-\mathtt{i}\omega hk)\,. \tag{2.2}$$

If g is b–band limited too, then

$$\int_{-\infty}^{\infty} f(\nu)\bar{g}(\nu)\,d\nu = h\sum_{k\in\mathbb{Z}} f(hk)\bar{g}(hk)\,. \tag{2.3}$$

Proof. The family of functions

$$\left\{u_k(\omega) = \frac{1}{\sqrt{2a}} \exp\Big(\mathtt{i}\pi\omega\frac{k}{a}\Big) : k \in \mathbb{Z}\right\}$$

is a complete orthonormal system in $L^2\big((-a,a);\mathbb{C}\big)$. That is, any function $\rho \in L^2\big((-a,a);\mathbb{C}\big)$ can be expanded into a Fourier series with respect to the system $\{u_k : k \in \mathbb{Z}\}$:

$$\rho = \sum_{k\in\mathbb{Z}} \rho_k u_k \,,$$

with

$$\rho_k = \int_{-a}^{a} \rho(\nu)\bar{u}_k(\nu)\,d\nu = \frac{1}{\sqrt{2a}} \int_{-a}^{a} \rho(\nu) \exp\Big(-\mathtt{i}\pi\nu\frac{k}{a}\Big)\,d\nu \,.$$

For ρ vanishing outside $(-a,a)$ we have

$$\rho_k = \frac{\sqrt{\pi}}{\sqrt{a}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(\nu) \exp\Big(-\mathtt{i}\pi\nu\frac{k}{a}\Big)\,d\nu = \sqrt{\frac{\pi}{a}}(\mathcal{F}^{-1}\rho)\Big(-\frac{\pi k}{a}\Big) \,.$$

This shows that

$$\rho(\omega) = \frac{\pi}{a}\frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} (\mathcal{F}^{-1}\rho)\Big(-\frac{\pi k}{a}\Big) \exp\Big(\mathtt{i}\pi\omega\frac{k}{a}\Big) \,.$$

Since f is assumed to be b–band limited and $h \le \pi/b$ we can identify ρ with $\mathcal{F}f$ and $a = \frac{\pi}{h}$ in the calculations above and obtain

$$(\mathcal{F}f)(\omega) = \frac{h}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} f(-hk)\exp(\mathtt{i}\omega hk) = \frac{h}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} f(hk)\exp(-\mathtt{i}\omega hk) \,,$$

which proves assertion (2.2).

Since f is b–band limited it follows from (2.2)

$$\begin{aligned}
f(t) &= \Big(\mathcal{F}^{-1}\Big((\mathcal{F}f)\chi_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right)}\Big)\Big)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mathcal{F}f)\chi_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right)}(\omega)\exp(\mathtt{i}\omega t)\,d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{h}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} f(hk)\exp(-\mathtt{i}\omega hk)\chi_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right)}(\omega)\exp(\mathtt{i}\omega t)\,d\omega \\
&= \frac{h}{2\pi} \sum_{k\in\mathbb{Z}} f(hk) \int_{-\infty}^{\infty} \exp\big(\mathtt{i}\omega(t-hk)\big)\chi_{\left(-\frac{\pi}{h},\frac{\pi}{h}\right)}(\omega)\,d\omega \\
&= \frac{h}{2\pi} \sum_{k\in\mathbb{Z}} f(hk) \int_{-\pi/h}^{\pi/h} \exp\big(\mathtt{i}\omega(t-hk)\big)\,d\omega
\end{aligned}$$

$$= \frac{h}{2\pi} \sum_{k\in\mathbb{Z}} f(hk) \frac{2\mathtt{i}\sin\bigl((hk-t)\pi/h\bigr)}{\mathtt{i}(hk-t)}$$
$$= \sum_{k\in\mathbb{Z}} f(hk)\,\mathrm{sinc}\Bigl((hk-t)\frac{\pi}{h}\Bigr)\,,$$

which proves assertion (2.1).

In order to prove (2.3) we use

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)(\mathcal{F}g)(t)\,dt &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty} g(\omega)\exp(-\mathtt{i}\omega t)\,d\omega\,dt\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\int_{-\infty}^{\infty} f(t)\exp(-\mathtt{i}\omega t)\,dt\,d\omega\\ &= \int_{-\infty}^{\infty} g(\omega)(\mathcal{F}f)(\omega)\,d\omega\end{aligned}$$

and

$$(\overline{\mathcal{F}g})(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{g}(t)\exp(\mathtt{i}\omega t)\,dt = (\mathcal{F}^{-1}\bar{g})(\omega)\,.$$

This shows

$$\begin{aligned}\int_{-\infty}^{\infty} (\mathcal{F}f)(\omega)(\overline{\mathcal{F}g})(\omega)\,d\omega &= \int_{-\infty}^{\infty} (\mathcal{F}f)(\omega)(\mathcal{F}^{-1}\bar{g})(\omega)\,d\omega\\ &= \int_{-\infty}^{\infty} f(t)\bigl(\mathcal{F}(\mathcal{F}^{-1}\bar{g})\bigr)(t)\,dt\\ &= \int_{-\infty}^{\infty} f(t)\bar{g}(t)\,dt\,.\end{aligned}$$

By (2.2) we have

$$(\mathcal{F}f)(\omega) = \sqrt{h}\sum_{k\in\mathbb{Z}} f(hk)\frac{\exp(-\mathtt{i}\omega hk)}{\sqrt{2\pi}}\sqrt{h}\,,$$
$$(\mathcal{F}g)(\omega) = \sqrt{h}\sum_{k\in\mathbb{Z}} g(hk)\frac{\exp(-\mathtt{i}\omega hk)}{\sqrt{2\pi}}\sqrt{h}\,,$$

and as the family of functions

$$\Bigl\{\frac{\exp(\mathtt{i}hk\omega)}{\sqrt{2\pi}}\sqrt{h} : k\in\mathbb{Z}\Bigr\}$$

is a complete orthonormal system on $L^2\big((-\frac{\pi}{h}, \frac{\pi}{h}); \mathbb{C}\big)$, we obtain

$$\int_{-\infty}^{\infty} f(t)\bar{g}(t)\, dt = \int_{-\infty}^{\infty} (\mathcal{F}f)(\omega)(\overline{\mathcal{F}g})(\omega)\, d\omega = h \sum_{k\in\mathbb{Z}} f(hk)\bar{g}(hk)\,. \qquad \square$$

Remark 2.3 (Nyquist condition). The condition $h \leq \frac{\pi}{b}$ is called *Nyquist condition*. It guarantees that a function f can be uniquely recovered from the discrete samples hk, $k \in \mathbb{Z}$, if the *sampling rate* h is less or equal to $\frac{\pi}{b}$, when b denotes the highest occuring frequency in the signal.

The Shannon Sampling Theorem 2.2 was established in 1948. Still nowadays it provides the basis of many applications, such as compact disc players. It is commonly accepted that the highest frequency recognized by human beings is about 20000Hz. Thus, according to the Shannon Sampling Theorem a sampling rate of 40000Hz is required. Compact disc players use a sampling rate of 44.100Hz, where the additional oversampling is used for error correction. One has to acknowledge that there is a different meaning of frequency for time signals and when we identify Fourier coefficients with frequency components. The n-th frequency component refers to the parts of the signal that have n-oscillations on the interval $(-\pi, \pi)$. That is it actually shows the angular frequency. A frequency of the time series of 20000Hz means an angular frequency of $b = 20000 * 2\pi$, which can be considered the band width of the music signal. The bandwidth b requires then a sampling frequency $h \leq \pi/b = 1/40000Hz$. ■

Remark 2.4 (Aliasing). *Aliasing* occurs if an analog signal is reconstructed from rarely sampled data, i.e., if $h > \frac{\pi}{b}$. To illustrate aliasing we consider a "wild–west" movie. A spinning wheel is recognized by human beings as moving forward if the rotation is relatively slow. If the rotational speed is increasing, then it is recognized to rotate "backwards". The camouflage is due to the fact that humans perceive (analog) sample pictures which are connected to the movie.

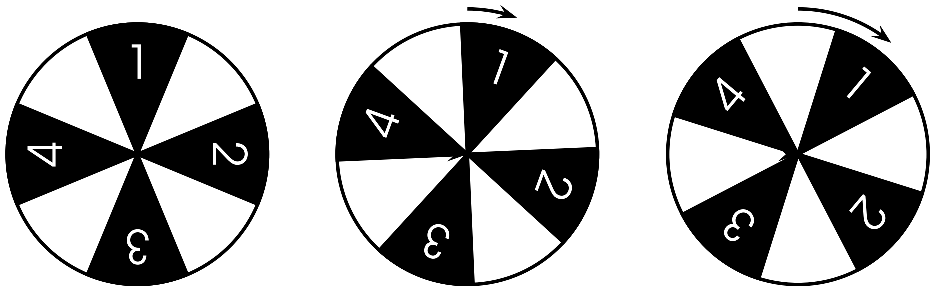

Figure 2.1: A slowly rotating wheel: actual and perceived sequences are corresponding.

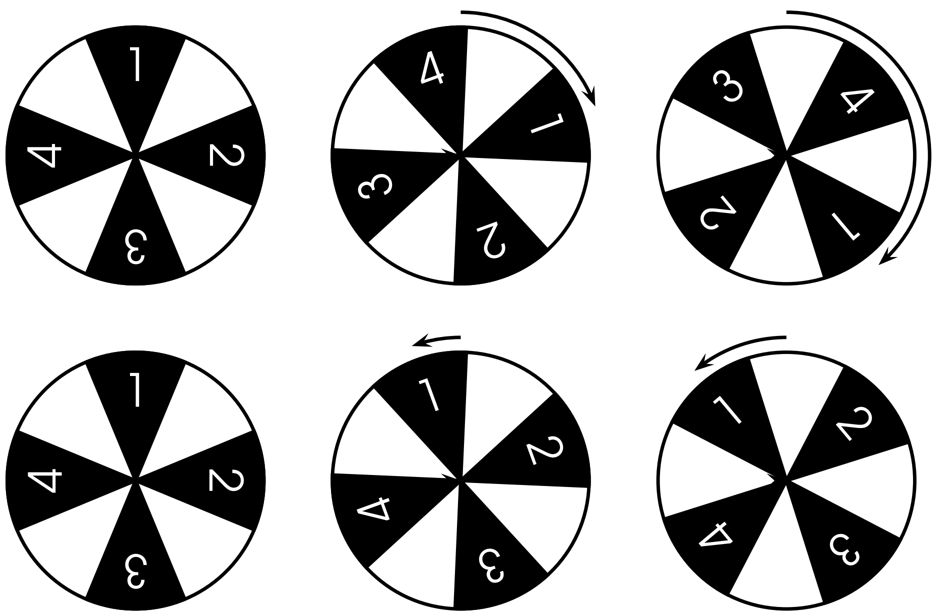

Figure 2.2: A fast rotating wheel: actual and perceived sequences are **not** corresponding.

Figures 2.1 and 2.2 show samples of a wheel rotating clockwise. The human perception system continuously connects corresponding sectors with minimal distance. Thus, in the case of a slow rotation the perceived rotation corresponds with the actual rotation (cf. Figure 2.1). If the rotational speed increases the rotation is recognized by humans to be in counter–clockwise direction (cf. Figure 2.2). ■

Chapter 3

Wavelets

In this chapter we introduce another transform widely used in image processing: the wavelet transform. We first present the basic idea behind wavelets by their motivation through *local* Fourier analysis. Finally we focus on the construction of compactly supported, orthonormal wavelet bases.

3.1 Windowed Fourier Transform

Definition 3.1. For a given function g the *windowed Fourier transform* of f is defined by

$$\phi(\omega, s) := \int_{-\infty}^{\infty} f(t)\overline{g(t-s)} \exp(-\mathtt{i}\omega t)\, dt = \langle f, g^{(\omega,s)} \rangle_{L^2(\mathbb{R};\mathbb{C})} \,, \tag{3.1}$$

with

$$g^{(\omega,s)}(t) := g(t-s) \exp(\mathtt{i}\omega t) \,, \tag{3.2}$$

whenever the integral exists. The function g is called the *window function*.■

The function $g^{(\omega,s)}$ depends on two arguments, the position in time (s) and the frequency (ω). The windowed Fourier transform $\phi(\omega, s)$ describes the *local* behavior of the function f at location s and for frequency ω.

Lemma 3.2. *The operator*

$$\begin{aligned} T : L^2(\mathbb{R};\mathbb{C}) &\to L^2(\mathbb{R}^2;\mathbb{C}) \\ f &\mapsto \phi(\omega, s) = \langle f, g^{(\omega,s)} \rangle_{L^2(\mathbb{R};\mathbb{C})} \end{aligned} \tag{3.3}$$

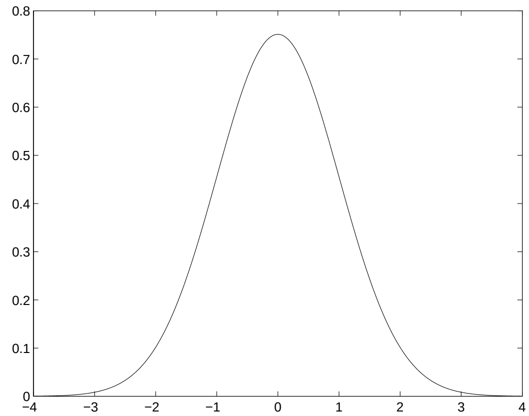

Figure 3.1: Example of a window function: $g(t) = \pi^{-1/4}\exp(-t^2/2)$.

is linear and, in addition, its operator norm is given by

$$\|T\| = \sqrt{2\pi}\|g\|_{L^2(\mathbb{R};\mathbb{C})} .$$

Proof. Using the notation $h(s) := \bar{g}(-s)$ and $j_\omega(s) := f(s)\exp(-\mathtt{i}\omega s)$, for each $\omega \in \mathbb{R}$ it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi(\omega,s)|^2 \, d\omega\, ds &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} f(t)\bar{g}(t-s)\exp(-\mathtt{i}\omega t)\, dt\right|^2 d\omega\, ds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} f(t)h(s-t)\exp(-\mathtt{i}\omega t)\, dt\right|^2 d\omega\, ds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(h * j_\omega)(s)|^2 \, d\omega\, ds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\mathcal{F}^{-1}\mathcal{F}(h * j_\omega)(s)\right|^2 d\omega\, ds \; .\end{aligned} \tag{3.4}$$

Using the isometry property of the Fourier transform (cf. Theorem 1.49)

$$\|\mathcal{F}\psi\|_{L^2(\mathbb{R};\mathbb{C})} = \|\psi\|_{L^2(\mathbb{R};\mathbb{C})} = \left\|\mathcal{F}^{-1}\psi\right\|_{L^2(\mathbb{R};\mathbb{C})} ,$$

we obtain for (3.4)

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi(\omega,s)|^2\,d\omega\,ds &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\mathcal{F}^{-1}\mathcal{F}(h*j_\omega)(s)\right|^2 d\omega\,ds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathcal{F}(h*j_\omega)(\hat\omega)|^2\,d\hat\omega\,d\omega \\ &= 2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathcal{F}h(\hat\omega)\mathcal{F}j_\omega(\hat\omega)|^2\,d\hat\omega\,d\omega\,.\end{aligned} \tag{3.5}$$

Now we calculate $\mathcal{F}j_\omega$ and $\mathcal{F}h$. From the definition of the Fourier transform it follows that

$$\mathcal{F}j_\omega(\hat\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\exp\bigl(-\mathtt{i}(\omega+\hat\omega)t\bigr)\,dt = \mathcal{F}f(\omega+\hat\omega)\,,$$

and

$$\begin{aligned}|\mathcal{F}h(\hat\omega)|^2 &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}\bar g(-t)\exp(-\mathtt{i}\hat\omega t)\,dt\right|^2 \\ &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}\bar g(t)\exp(\mathtt{i}\hat\omega t)\,dt\right|^2 \\ &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} g(t)\exp(-\mathtt{i}\hat\omega t)\,dt\right|^2 \\ &= |\mathcal{F}g(\hat\omega)|^2\,.\end{aligned}$$

Using these two identities we can further transform (3.5) and obtain

$$\begin{aligned}&2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathcal{F}h(\hat\omega)\mathcal{F}j_\omega(\hat\omega)|^2\,d\omega\,d\hat\omega \\ &= 2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathcal{F}g(\hat\omega)|^2\,|\mathcal{F}f(\omega+\hat\omega)|^2\,d\omega\,d\hat\omega \\ &= 2\pi\int_{-\infty}^{\infty} |\mathcal{F}g(\hat\omega)|^2\int_{-\infty}^{\infty} |\mathcal{F}f(\omega+\hat\omega)|^2\,d\omega\,d\hat\omega \\ &= 2\pi\int_{-\infty}^{\infty} |\mathcal{F}g(\hat\omega)|^2\,d\hat\omega\,\int_{-\infty}^{\infty} |\mathcal{F}f(\omega)|^2\,d\omega \\ &= 2\pi\|g\|^2_{L^2(\mathbb{R};\mathbb{C})}\|f\|^2_{L^2(\mathbb{R};\mathbb{C})}\,.\end{aligned}$$

This shows the assertion. □

In application the signal is analyzed with the windowed Fourier transform by discretizing the phase space with the points

$$(m\omega_0\,, ns_0)_{(m,n)\in\mathbb{Z}^2}\,.$$

As a consequence, the window function in a discrete point (m,n) is defined by

$$g_{m,n}(t) := g^{(m\omega_0,ns_0)}(t) = g(t-ns_0)\exp(\mathtt{i}m\omega_0 t)\,, \tag{3.6}$$

which motivates the following definition:

Definition 3.3. The *discrete windowed Fourier transform* is defined as the mapping

$$\begin{aligned} T^d : L^2(\mathbb{R};\mathbb{C}) &\to l^2(\mathbb{Z}^2). \\ f &\mapsto c_{m,n}[f] := \langle f, g_{m,n}\rangle_{L^2(\mathbb{R};\mathbb{C})} = \phi(m\omega_0, ns_0)\,. \end{aligned}$$

The terms $c_{m,n}[f]$ are called the *windowed Fourier coefficients of f*. ■

Notice, in comparison to T the operator T^d maps a function f into the space of square summable vectors.

To reconstruct f from the windowed Fourier coefficients $\{c_{m,n}[f]\}$ we have to know that T^d is injective. In addition, for the stable inversion we have to know that the inverse of T^d is bounded.

In many applications it is more efficient to rely on a logarithmic scale of the frequency: In music a change by one octave corresponds to a variation of the frequency by a factor 2. In the sequel we investigate a logarithmic frequency discretization with an appropriate constant k_0:

$$\omega_m = \frac{k_0}{2^m}\,.$$

From sampling theory, that is from the Shannon Sampling Theorem (see Theorem 2.2), it follows that for the reconstruction of a b–band limited signal, that is a signal with angular frequency range $[-b,b]$, a sampling rate according to the Nyquist–condition

$$\Delta t \le \frac{\pi}{b}$$

is required. More precisely, a b–band limited signal can be uniquely recovered from the samples of the signal at time $k\Delta t$, $k\in\mathbb{Z}$. This result implies that a finer grid is only required for the sampling of the high frequency components of the signal. Thus we use a frequency dependent sampling of the time domain with sampling distance

$$\Delta t = C\frac{\pi}{\omega} \quad \text{where} \quad C < 1\,.$$

Both ideas together, the use of a logarithmic scale and taking into account the Nyquist–condition for the different frequency components of the signal, suggest the use of the following (time–frequency adapted) grid:

$$\omega_m = \frac{k_0}{2^m}, \quad t_{m,n} = n\frac{C\pi 2^m}{k_0} = \frac{C\pi n}{\omega_m}\,.$$

For the realization of such a sampling with a windowed Fourier like transform one requires a window function with a variable time domain. Note that $g_{m,n}$, as in (3.6) is defined via a product of two functions, each depending only on m and on n, respectively. In particular, m and n are independent and, therefore can not be used in a time-frequency adapted framework.

3.2 Wavelet Transform

The basic idea of the wavelet transform consists in using instead of the window function g and the family $g^{(\omega,s)}$, as in the windowed Fourier transform, an analyzing *wavelet* ψ and its associated family of functions $\psi^{(a,b)}$, defined for $a \neq 0$ and arbitrary $b \in \mathbb{R}$, by

$$\psi^{(a,b)}(t) := |a|^{-\frac{1}{2}}\,\psi\left(\frac{t-b}{a}\right)\,, \quad t \in \mathbb{R}\,. \tag{3.7}$$

The parameters (a, b) can be again considered as elements of a phase space. However, in the window functions $\psi\left(\frac{t-b}{a}\right)$ the parameters a and b are related. The factor $|a|^{-\frac{1}{2}}$ is introduced to guarantee that the energy is independent of a and b, i.e.,

$$\int_{-\infty}^{\infty} \left|\psi^{(a,b)}(t)\right|^2 dt = \int_{-\infty}^{\infty} |\psi(t)|^2\, dt\,.$$

For fixed b the support of $\psi^{(a,b)}$ is monotonically increasing in a. In other words, the smaller a is, the smaller details of the function to be investigated with wavelets can be observed.

In analogy to the discrete windowed Fourier transform we introduce the discrete wavelet transform:

Definition 3.4. Let $a_0, b_0 \in \mathbb{R}$ with $a_0 \neq 0$. For $j, n \in \mathbb{Z}$ we define the discrete family of wavelet functions by

$$\psi_{j,n}(t) = \psi^{(a_0^j, nb_0 a_0^j)}(t) = a_0^{-\frac{j}{2}} \psi(a_0^{-j} t - nb_0) \,. \tag{3.8}$$

Note that this corresponds to setting $a := a_0^j$ and $b := nb_0 a_0^j$ in (3.7). The discrete wavelet coefficients are defined as

$$c_{j,n}^w[f] := \langle f, \psi_{j,n} \rangle_{L^2(\mathbb{R};\mathbb{C})} = a_0^{-\frac{j}{2}} \int_{-\infty}^{\infty} \bar{\psi}(a_0^{-j} t - nb_0) f(t) \, dt \,.$$

■

In the following we investigate under which conditions it is possible to uniquely reconstruct f from the discrete wavelet coefficients $\{c_{j,n}^w[f]\}$. In other words, we study, under which condition it is possible to invert the operator

$$\begin{aligned} T^w : L^2(\mathbb{R};\mathbb{C}) &\to l^2(\mathbb{Z}^2) \\ f &\mapsto (c_{j,n}^w[f])_{(j,n)\in\mathbb{Z}^2} \,. \end{aligned} \tag{3.9}$$

From algebra we know that if the wavelet family $\{\psi_{j,n}\}$ is an orthonormal basis of $L^2(\mathbb{R};\mathbb{C})$ the operator T^w is invertible and

$$f = \sum_{(j,n)\in\mathbb{Z}^2} c_{j,n}^w[f] \psi_{j,n} = (T^w)^{-1}(c_{j,n}^w[f]) \,.$$

The existence of a complete, compactly supported, orthonormal wavelet family has been an open problem for a long time. We will discuss the construction below.

The existence of an orthonormal wavelet family is sufficient but not necessary for the invertibility of the operator T^w. Bounded invertibility is also guaranteed if there exist positive numbers $0 < A \leq B < \infty$ such that

$$A\|f\|_{L^2(\mathbb{R};\mathbb{C})}^2 \leq \sum_{(j,n)\in\mathbb{Z}^2} \left| c_{j,n}^w[f] \right|^2 \leq B\|f\|_{L^2(\mathbb{R};\mathbb{C})}^2 \,. \tag{3.10}$$

Definition 3.5. A family $\{\psi_{j,n} : j, n \in \mathbb{Z}\}$ satisfying (3.10) is called a *frame*. Each frame has a *dual frame* $\{\tilde{\psi}_{j,n} : j, n \in \mathbb{Z}\}$ satisfying

$$f = \sum_{(j,n)\in\mathbb{Z}^2} c_{j,n}^w[f] \tilde{\psi}_{j,n} \,. \tag{3.11}$$

■

3.3 Orthonormal Wavelets on $\mathbb{R}$

The construction of compactly supported, orthonormal wavelet bases for $L^2(\mathbb{R};\mathbb{C})$ dates back to Daubechies [1, 2].

The construction of wavelets is related to the construction of a *scaling function* ϕ such that for fixed $m \in \mathbb{Z}$ the functions

$$\phi_{m,k}(x) := 2^{-m/2}\phi(2^{-m}x - k)\,, \quad k \in \mathbb{Z}\,,$$

are orthonormal with respect to $L^2(\mathbb{R};\mathbb{C})$.[1] Moreover, for $m \in \mathbb{Z}$ the spaces

$$V_m := \overline{\texttt{span}\,\{\phi_{m,k} : k \in \mathbb{Z}\}}$$

constitute a *multiresolution analysis* for $L^2(\mathbb{R};\mathbb{C})$, that is

$$V_m \subset V_{m-1}\,, \quad m \in \mathbb{Z}\,,$$

and

$$\bigcap_{m\in\mathbb{Z}} V_m = \{0\} \text{ and } \overline{\bigcup_{m\in\mathbb{Z}} V_m} = L^2(\mathbb{R};\mathbb{C})\,.$$

The *wavelet spaces* W_m are the orthogonal complements of V_m in V_{m-1}, that is

$$W_m := V_m^{\perp} \cap V_{m-1}\,.$$

One defines the wavelet ψ such that the functions

$$\psi_{m,k}(x) := 2^{-m/2}\psi(2^{-m}x - k)\,, \quad k \in \mathbb{Z}\,,$$

are an orthonormal basis of W_m. Since both V_m and W_m are contained in V_{m-1} the scaling function ϕ must satisfy the *dilation equation*

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k \phi(2x - k)\,, \tag{3.12}$$

where the sequence $\{h_k\}$ is known as the *filter sequence* and satisfies constraints stated below. Correspondingly, the wavelet ψ satisfies

$$\psi(x) = \sum_{k\in\mathbb{Z}} g_k \phi(2x - k)\,, \tag{3.13}$$

[1] We only consider scaling factors and wavelets with a scaling factor $a = 2$ and dilation $b = 1$, as they are most widely used. Nevertheless other choices are possible.

where the canonical setting

$$g_k = (-1)^k h_{1-k}$$

guarantees that the functions $\{\psi_{m,k} : k \in \mathbb{Z}\}$ are an orthonormal basis.

We emphasize that the support of $\phi_{m,k}$ and $\psi_{m,k}$ is double as large as the support of $\phi_{m-1,k}$ and $\psi_{m-1,k}$. Thus, high frequency components (small details) are captured by functions $\phi_{m,k}$ and $\psi_{m,k}$ with small integers.

Daubechies [1] established conditions on the filter sequence $\{h_k\}$ in order to ensure

1. the dilation equation (3.12) has a solution $\phi \in L^2(\mathbb{R};\mathbb{C})$,
2. with $\operatorname{supp}(\phi) = [-N+1, N]$ for a given integer N,
3. and that for fixed m the functions $\phi_{m,k}$ are orthogonal,
4. with the property that polynomials up to degree $N-1$ can be represented as linear combinations of $\phi_{m,k}$.

In the following we derive the necessary conditions for $\{h_k\}$ from these postulates.

Lemma 3.6. *Compact support of ϕ in $[-N+1, N]$ is ensured by*

$$h_k = 0\,, \text{ for } k < 1-N \text{ or } k > N\,. \tag{3.14}$$

Proof. Notice that if $\phi(x)$ has support in $[-N+1, N]$ the function $\phi(2x-k)$ has support in $\left[\frac{1-N+k}{2}, \frac{N+k}{2}\right]$, and vice versa. The dilation equation therefore requires that $h_k = 0$ if and only if k does not satisfy

$$\left[\frac{1-N+k}{2}, \frac{N+k}{2}\right] \subseteq [-N+1, N]\,.$$

Thus $h_k = 0$ for $k < -N+1$ or $k > N$. □

Lemma 3.7. *A necessary condition for the existence of a solution of the dilation equation (see (3.12)) is*

$$\sum_{k=1-N}^{N} h_k = 2\,. \tag{3.15}$$

Proof. Integrating (3.12) it follows that

$$\int_{-\infty}^{\infty} \phi(x)\,dx = \sum_{k\in\mathbb{Z}} h_k \int_{-\infty}^{\infty} \phi(2x-k)\,dx = \frac{1}{2}\sum_{k\in\mathbb{Z}} h_k \int_{-\infty}^{\infty} \phi(x)\,dx\,,$$

which implies the assertion. □

Lemma 3.8. *Orthonormality of the translates of ϕ, that is*

$$\int_{-\infty}^{\infty} \phi(x)\phi(x-l)\,dx = \delta_{0,l}\,,$$

implies[2] *that for $l = 0,\ldots,N-1$*

$$\sum_{k=1-N}^{N} h_k h_{k-2l} = 2\delta_{0,l}\,. \tag{3.16}$$

Proof. This follows again from the dilation equation (3.12) and substitution. If ϕ and the dilates are orthonormal, then

$$\begin{aligned}\delta_{0,l} = \int_{-\infty}^{\infty} \phi(x)\phi(x-l)\,dx &= \sum_{k,j\in\mathbb{Z}} h_k h_j \int_{-\infty}^{\infty} \phi(2x-k)\phi\big(2(x-l)-j\big)\,dx \\ &= \frac{1}{2}\sum_{k,j\in\mathbb{Z}} h_k h_j \int_{-\infty}^{\infty} \phi(x)\phi(x+k-2l-j)\,dx \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}} h_k h_{k-2l}\,,\end{aligned}$$

which proves the assertion. □

Lemma 3.9. *If polynomials up to degree $N-1$ are exactly representable in V_0, then for $l = 0,\ldots,N-1$*

$$\sum_{k=1-N}^{N} (-1)^k h_{1-k} k^l = 0\,. \tag{3.17}$$

[2] Of course it is sufficient to consider this only for the zero-th scale.

Proof. The wavelet space W_0 is orthonormal to the space V_0. Since V_0 contains all polynomials up to degree $N-1$ and $\psi \in W_0$, it follows that for $l = 0, \ldots, N-1$

$$\int_{-\infty}^{\infty} x^l \psi(x)\, dx = 0\,.$$

Thus, for $l = 0, \ldots, N-1$ we have

$$\begin{aligned}
0 = \int_{-\infty}^{\infty} x^l \psi(x)\, dx &= \sum_{k=1-N}^{N} (-1)^k h_{1-k} \int_{-\infty}^{\infty} x^l \phi(2x-k)\, dx \\
&= \frac{1}{2} \sum_{k=1-N}^{N} (-1)^k h_{1-k} \int_{-\infty}^{\infty} \left(\frac{y+k}{2}\right)^l \phi(y)\, dy \\
&= \frac{1}{2} \sum_{k=1-N}^{N} (-1)^k h_{1-k} \left(\sum_{n=0}^{l} \binom{l}{n} k^{l-n} \int_{-\infty}^{\infty} y^n \phi(y)\, dy \right).
\end{aligned}$$

In particular for $l = 0$ we obtain

$$0 = \int_{-\infty}^{\infty} \psi(x)\, dx = \frac{1}{2} \sum_{k=1-N}^{N} (-1)^k h_{1-k} \int_{-\infty}^{\infty} \phi(y)\, dy$$

and thus

$$0 = \sum_{k=1-N}^{N} (-1)^k h_{1-k}\,.$$

For $l = 1$ we have

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty} x\psi(x)\, dx \\
&= \frac{1}{2} \sum_{k=1-N}^{N} (-1)^k h_{1-k} k \int_{-\infty}^{\infty} \phi(y)\, dy + \frac{1}{2} \int_{-\infty}^{\infty} y\phi(y) dy \underbrace{\sum_{k=1-N}^{N} (-1)^k h_{1-k}}_{=0 \text{ because of } l=0}\,.
\end{aligned}$$

Thus,

$$0 = \sum_{k=1-N}^{N} (-1)^k k h_{1-k}\,.$$

Now, by induction with respect to l the assertion can be proven. □

Example 3.10. In case of $N = 1$ the solution is trivial: the Haar wavelet defined by $h_0 = h_1 = 1$ is the only possible solution. For $N = 2$ one is interested in all common roots of the five polynomials

$$\begin{aligned} -2 + x_1 + x_2 + x_3 + x_4 \,, \\ -2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 \,, \\ x_1 x_3 + x_2 x_4 \,, \\ x_1 - x_2 + x_3 - x_4 \,, \\ 2x_1 - x_2 + x_4 \,, \end{aligned} \tag{3.18}$$

in the four variables

$$x_1 = h_{-1},\ x_2 = h_0,\ x_3 = h_1,\ \text{and } x_4 = h_2 \,.$$

There exists two solutions for the filter coefficients:

$$(x_1, x_2, x_3, x_4) = \left(\frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4} \right) \,,$$

and

$$(x_1, x_2, x_3, x_4) = \left(\frac{1-\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{1+\sqrt{3}}{4} \right) \,. \qquad \blacksquare$$

In general it is not possible to provide a closed form of the scaling function. Instead the function values have to be determined recursively. For this purpose we recall the dilation equation:

$$\phi(x) = h_{1-N}\phi(2x + N - 1) + h_{2-N}\phi(2x + N - 2) + \ldots + h_N\phi(2x - N) \,.$$

We insert only integer numbers $j = -N + 1, \ldots, N$ in this equation and find

$$\begin{aligned} \phi(1-N) &= h_{1-N}\phi(1-N) \,, \\ \phi(2-N) &= h_{3-N}\phi(1-N) + h_{2-N}\phi(2-N) + h_{1-N}\phi(3-N) \,, \\ &\vdots \\ \phi(N-1) &= h_N\phi(N-2) + h_{N-1}\phi(N-1) + h_{N-2}\phi(N) \,, \\ \phi(N) &= h_N\phi(N) \,. \end{aligned}$$

In matrix form this system of equation reads as follows:

$$\underbrace{\begin{bmatrix} h_{1-N} & 0 & 0 & \dots & 0 & 0 & 0 \\ h_{3-N} & h_{2-N} & h_{1-N} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & h_{N-2} & h_{N-3} & h_{N-4} \\ 0 & 0 & 0 & \dots & h_N & h_{N-1} & h_{N-2} \\ 0 & 0 & 0 & \dots & 0 & 0 & h_N \end{bmatrix}}_{\mathcal{M}} \underbrace{\begin{bmatrix} \phi(1-N) \\ \phi(2-N) \\ \vdots \\ \vdots \\ \phi(N-2) \\ \phi(N-1) \\ \phi(N) \end{bmatrix}}_{\Phi} = \begin{bmatrix} \phi(1-N) \\ \phi(2-N) \\ \vdots \\ \vdots \\ \phi(N-2) \\ \phi(N-1) \\ \phi(N) \end{bmatrix}.$$

This shows that the vector Φ containing the values of the scaling function at integer values is an eigenvector according to the eigenvalue $\lambda = 1$ of the matrix $\mathcal{M}$. The solution of the eigenvalue problem

$$(\mathcal{M} - I)\Phi = 0\,,$$

is unique, up to a change of sign, under the normalization condition

$$\int_{-\infty}^{\infty} \phi(x)\,dx = 1\,.$$

As a consequence of the dilation equation followed by a substitution the normalization condition of ϕ implies that

$$1 = \int_{-\infty}^{\infty} \phi(x)\,dx = \frac{1}{2}\sum_{k\in\mathbb{Z}} h_k \int_{-\infty}^{\infty} \phi(y)\,dy = \sum_{k\in\mathbb{Z}} \phi(k)\,.$$

Once Φ has been determined, the function values of ϕ at the dyadic points $\{i/2^n : i, n \in \mathbb{Z}\}$ can be determined recursively from the dilation equation:

$$\phi\left(\frac{x}{2}\right) = \sum_{k=-\infty}^{\infty} h_k \phi(x-k)\,.$$

Chapter 4

Principles of Lossy Data Compression

A lossy data compression algorithm consists of three successive steps (cf. Figure 4.1):

1. *transform* to represent the data in a compact form,
2. *quantization* to eliminate *non essential* information, and
3. *entropy coding* for efficiently storing quantized data.

Compression algorithms based on the DCT calculate the Fourier cosine coefficients. The first half of the coefficients contain the essential information. The coefficients of the second half are quantized. The first half of coefficients can be decomposed again with the DCT and the latter coefficients can be quantized again. This steps can be performed iteratively. Finally all collected coefficients are entropy encoded, for instance with a *Huffman coder*.

Transform: The most widely used transforms for data compression are wavelet transform and the DCT.

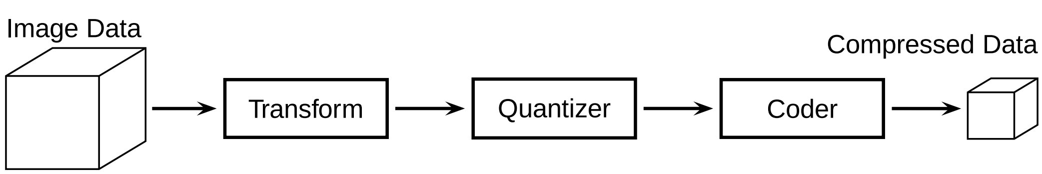

Figure 4.1: Schematic representation of the wavelet compression algorithm.

1. The DCT is the base of the JPEG compression standard. JPEG divides the image into distinct 8×8 or 16×16 blocks. The DCT is performed for each block. The coefficients in each block are quantized. The quantized data are encoded before transmission.

   The principle of lossy data compression is that the low frequency components contain the essential information which has to be stored accurately. The high frequency components can be stored with low accuracy.

   In Figure 4.2 we have highlighted the principle of the JPEG algorithm. It is readily seen that many of the DCT coefficients can be set to zero without severely affecting the reconstruction. Setting many coefficients to zero implies high compression ratios. At high compression ratios the JPEG algorithm produces block artefacts.

2. The wavelet transform is the base of *JPEG2000*.

4.1 Data Analysis with Wavelets

Given a function $f \in L^2(\mathbb{R};\mathbb{C})$ we define its projections P_m, Q_m onto the wavelet and scaling function subspaces W_m and V_m. From the orthogonality of V_m and W_m it follows that

$$P_m f = P_{m+1} f + Q_{m+1} f \,. \tag{4.1}$$

Thus $P_{m+1}f$ can be interpreted as the *low frequency* components of $P_m f$ and $Q_{m+1}f$ are the *high frequency* components. $P_{m+1}f$ is the projection on V_{m+1}, that is the space defined by the scaling function of scale m, and this projection is interpreted as *low-pass filtering*. The high frequency components are described by wavelets. Thus this is *high-pass filtering*. Compression of a wavelet by a factor two in the space coordinate means a scaling in frequency by a factor 1/2. If ψ represents a frequency range $[0.5, 1.0]$Hz, then $\psi(2x)$ represents a frequency range $[1.0, 2.0]$Hz. Having available a wavelet with good frequency and time/space localization properties, it can be used to recognize certain features in a signal/image.

In data compression wavelets are used to store the wavelet coefficients instead of an image itself. We describe the approach for a two-dimensional image. An image can be considered a discrete sampling of a function of

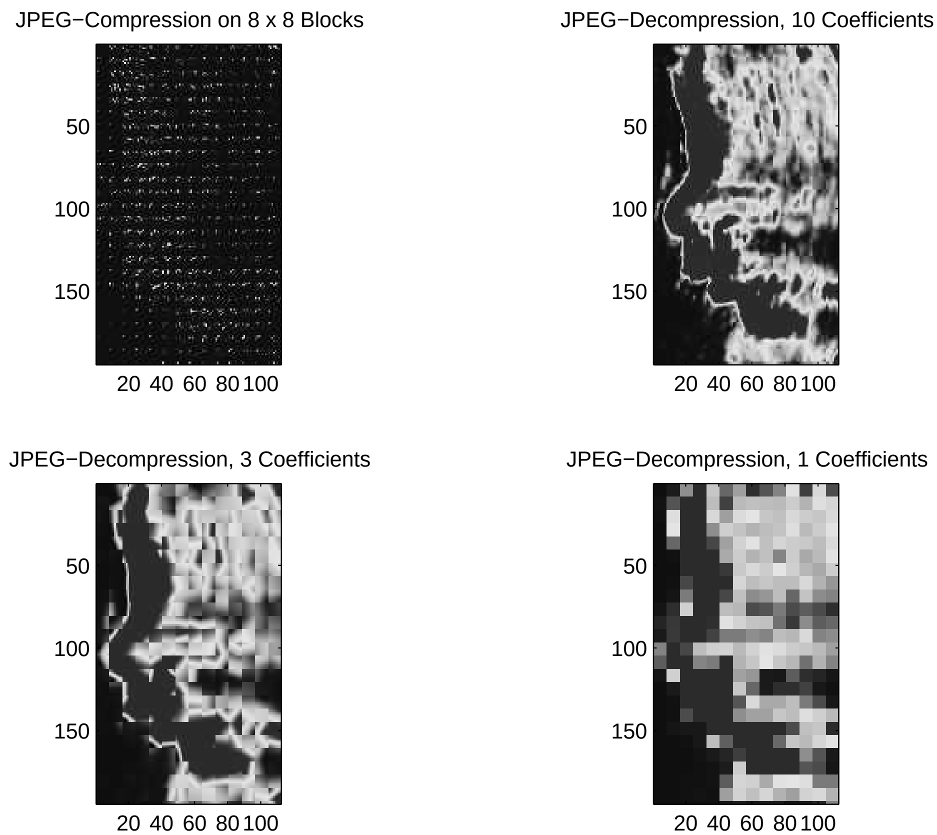

Figure 4.2: In the JPEG algorithm 8×8 blocks are DCT transformed. The top left image shows the DCT transformed data. In the top right image only 10 coefficients in each 8×8 block are left unaltered, all others are set to zero. In the bottom left and right image only 3 and 1 coefficients are not set to zero, respectively.

two variables. We think of a function f defined on the interval $[0,1] \times [0,1]$ (the image), which is periodically continued. Actually typically a discrete image is assumed, which we denote by $(f_{k,l})$ and where we identify the values of a function f. To be precise, we assume that $f_{k,l} = f(2^{-m}k, 2^{-m}l)$ for $k, l = 1, \dots, 2^m$. That is, we assume an image size of $2^m \times 2^m$ pixels.

We recall that a two-dimensional wavelet decomposition of f is a tensor product[1]:

$$f(x,y) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \eta_{j,k,l} \psi_{j,k}(x) \psi_{j,l}(y) \ . \tag{4.2}$$

In compression, the first step is to consider $(f_{k,l})$ exactly representable by a smallest scale m, typically this is determined by the pixel size:

$$f(x,y) \sim P_m f(x,y) = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \gamma_{m,k,l} \phi_{m,k}(x) \phi_{m,l}(y) \ . \tag{4.3}$$

In applications the scaling function composition does not have to be calculated explicitly, since the scaling function approximates the δ–function, and thus it is justified to identify the image with the scaling function coefficients: That is, on the finest scale we have

$$\gamma_{m,k,l} \sim f_{k,l} = f(2^{-m}k, 2^{-m}l) \ . \tag{4.4}$$

4.2 Mallat Algorithm

The Mallat algorithm is used to calculate $P_{m+1}f + Q_{m+1}f$ from $P_m f$ (cf. (4.1)). For the sake of simplicity of presentation, we outline the algorithm only for the one-dimensional case. In two dimension it is similar, since we work with a tensor ansatz, but technically much more involved. Moreover, we present the algorithm just for the scale $m = 0$. For all other indices it is analog by a simple renumbering. We use on $m = 0$ the identity (4.4), which in the one-dimension case are

$$\gamma_k^0 := \gamma_{0,k} \sim f_k \ .$$

[1] We use here representations via wavelets and there are no scaling function components.

Then, we use the dilation equation (3.12) identity (note $\gamma_1 := (\gamma^1_k)_{k\in\mathbb{Z}})$ are the coefficients at a larger scale)

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \gamma^1_k \phi_{1,k}(x) &= \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \gamma^1_k \phi\left(\frac{1}{2}x - k\right) \\
&= \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \gamma^1_k \sum_{l\in\mathbb{Z}} h_l \phi\left(2\left(\frac{1}{2}x - k\right) - l\right) \\
&= \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \gamma^1_k \sum_{l\in\mathbb{Z}} h_l \phi_{0,2k+l}(x) \\
&= \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \gamma^1_k \sum_{\eta\in\mathbb{Z}} h_{\eta-2k} \phi_{0,\eta}(x) \\
&= \sum_{\eta\in\mathbb{Z}} \underbrace{\left(\frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \gamma^1_k h_{\eta-2k}\right)}_{=\gamma^0_\eta} \phi_{0,\eta}(x) \ .
\end{aligned}$$

In matrix form this reads as:

$$\gamma^0 = \mathcal{H}\gamma^1 \,,$$

where

$$\mathcal{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & h_{1-N} & h_{3-N} & h_{5-N} & \dots & 0 \\ 0 & \dots & 0 & h_{2-N} & h_{4-N} & h_{6-N} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} .$$

Sometimes it is also written as a convolution with extended vectors:

$$\gamma^0 = \mathcal{H}\gamma^1_e =: \mathcal{H}(\dots, \gamma^1_1, 0, \gamma^1_2, 0, \dots)^T \,,$$

where

$$\mathcal{H}_e = \frac{1}{\sqrt{2}} \begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & h_{1-N} & h_{2-N} & h_{3-N} & 0 \\ 0 & \dots & h_{1-N} & h_{2-N} & h_{3-N} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} .$$

In other words

$$\gamma^0 = h * \gamma^1_e \ .$$

The matrix is periodically extended in the same way as the image is. In practical application it has to be truncated and then at the boundaries, depending on the form of extension of the image, other lines occur. Note also the special structure of the matrix: Each line corresponds to a convolution, however, the convolution corresponds to shifts of two entries. Taking into account that the wavelet coefficient are just the scaling functions coefficients in reverse order, modulo a sign, the calculation of the wavelet coefficients is identical.

The fast implementation of the Mallat algorithm is similar to FFT methods, and thus require such Pixel size numbers 2^m of the original matrix. However, such a matrix can always be obtained by zero padding. The fast implementation of matrix multiplications $\mathcal{H}\gamma^1$ is by the Mallat algorithm.

Chapter 5

Morphological Image Processing

In this lecture we concentrate on some basic morphological image processing. We consider images as two dimensional signals, and therefore products of Fourier, wavelets methods, respectively, can be used for the image analysis. However these methods have preferred axial directions. The following kind of techniques overcome this drawback. More sophisticated filtering techniques, such as partial differential equations, cannot be considered for time reasons. The standard reference to Morphological Image Processing is [5].

5.1 What is mathematical morphology?

Morphology is concerned with *theory and analysis of image objects*. Thereby properties of objects such as form, intensity, texture are analyzed. Important properties thereby are neighboring relations in the image. The axiomacy is based on invariance under gray value transformations, and as a consequence Morphology is *nonlinear*. The theory of morphology is based on sets and topology. The numerical implementation can be performed with efficient algorithms. It is one of the most efficient techniques in image processing.

5.2 Discretization

We assume that the grey valued image can be described by a continuous function $u : \mathbb{R}^2 \to \mathbb{R}$. We assume that u is only given by discrete samples, that are function values at nodes $x_i, i = 1, \ldots, N$:

$$u_i := u(x_i).$$

Typically we take a uniform grid with quadratic cells. Here and in the following we assume that u and (u_i) is non-negative. The largest intensity of u is denoted by $t_{\max}$.

5.3 Grey valued images

The subgraph of u at level l is the set

$$\mathcal{G}_l(u) := \{x : 0 \leq u(x) \leq l\}$$

We define the levels of the images by the following operations

$$(T_{[t_i,t_j]}u)(x) = \begin{cases} 1 & t_i \leq u(x) \leq t_j \\ 0 & \text{else} \end{cases}$$

Histograms of grey valued images deliver important information on images, which can be used for contrast improvement, segmentation and object recognition. Thereby for every grey value $t \in W = \{0, \ldots, t_{max}\}$ the number $H_u(t)$ of pixels are counted which share the same intensity

$$H_u : W \to \mathbb{N}_0.$$

Scaling of the histogram by the number of pixels of the image, that is by $\frac{1}{\#}$ can be considered an approximation of a statistical distribution of the intensities of the sample u.

5.4 Basic morphological operations

We differ between

- point operations: $\psi(u)(x) = \tilde{\psi}(u(x))$, where $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ is a transformation of the intensities,
- local (neighborhood-) operations, where $\psi(u)(x)$ is determined from the values of the sample $u(y)$ in a neighborhood of x, and

- global operations, such as translations, rotations and reflections.

Some important auxiliary operations are:

- Pointwise maximum of two images

$$(u \wedge v)(x) = \max\{u(x), v(x)\} \ .$$

- Pointwise minimum

$$(u \vee v)(x) = \min\{u(x), v(x)\} \ .$$

- Complements.

$$(Cu)(x) = u^C(x) = t_{\max} - u(x)$$

- Scaling of grey values:

$$\psi_c(u)(x) = P_W(c \cdot u(x))$$

  with some $c > 0$. Here P_W is the projection onto some intensity set W, defined by

$$P_W(u) = \min(\max(\lfloor u \rfloor, 0), t_{\max})$$

  and $\lfloor u \rfloor$ denote the largest number less than u, which is contained in W. For instance the largest integer value smaller that u.

- Binarisation:

$$\psi_c(u)(x) = \begin{cases} 1 & \text{falls } u(x) \geq c \\ 0 & \text{sonst} \end{cases}$$

  with $c \in [0, t_{max}]$. This is a simple form of segmentation.

Examples of local operations are for example the discrete convolution

$$\psi(u)(x) = \sum_{y \in \mathbb{Z}^2} u(x-y)K(y),$$

where K is an appropriate kernel function. A simple example of K is by averaging over five pixels:

$$K(x,y) = \begin{cases} \frac{1}{5} & \text{falls } |x| + |y| \leq 1 \\ 0 & \text{sonst} \end{cases}$$

When discretized, K is typically described by a matrix, sometimes also called ($mask$). For the example above we have:

$$\begin{pmatrix} 0 & \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & 0 \end{pmatrix} .$$

Convolution can be used for filtering (denoising).

5.4.1 Erosion and Dilation

Basic morphological operations are *erosion* and *dilatation*, which are defined via *structural elements*. A structural element is a small set B_x with a center

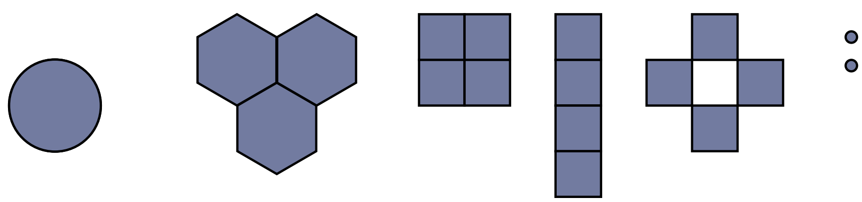

Figure 5.1: various structural elements.

of reference x. It is used to analyze/manipulate the image in a neighborhood of x.

Examples of commonly used structural elements are:

- Unit sphere $B_1(0)$ with center of reference 0.
- Rectangle $R_{a,b}$ with side lengths a,b and center of reference 0.

In the following let X be an object (set of pixels), for instance the sublevel set of a grey valued image.

For binary images (or image objects), erosion is defined as

$$\epsilon_B(X) := \bigcap_{b \in B} X_{-b} = \bigcap_{b \in B} \{X - b\} ,$$

that is the intersection of translations according to a structural element B. For grey valued images it is defined as the pointwise minimum of translations

$$\epsilon_B(f) := \wedge_{b \in B} f_{-b}$$

Figure 5.2: Erosion of a set X (yellow) by B (grey).

or in other words

$$\epsilon_B(f)(x) := \min_{b \in B} f(x+b) \ .$$

Dilation is the *"dual" of erosion*: For binary images it is defined as the union of translation

$$\delta_B(X) := \bigcup_{b \in B} X_{-b} \ .$$

For grey values images it is defines as the pointwise maximum of translations

$$\delta_B(f) = \max_{b \in B} f(x+b)$$

Dilation and erosion satisfy the following properties:

- The are dual with respect to complementation, that is,

$$\epsilon_B = C \delta_B C \ .$$

- Erosion shrinks objects and expands the background. Dilation expands objects and shrinks the background.

- Erosion and dilation are monotone transformations:

$$f \leq g \quad \Rightarrow \quad \epsilon_B(f) \leq \epsilon_B(g), \ \delta_B(f) \leq \delta_B(g)$$

All morphological filter operations are composed from erosion and dilation. For instance, the composition of $\delta_B \epsilon_B$ erases small objects. Thus the composition is in fact a filtering technique.

5.5 The most important Filter

in morphology is the *median*. It relies on sorting the intensities in a neighborhood, and taking the intensity value of the sort with the middle index.

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