Active Contours (Solution via gradient descent)

Lect 4 (1)

$$E_f(\Gamma) = \alpha \int_\Gamma d\Gamma + \beta \int_\Gamma \underbrace{h(|Df|)}_{=:\,\phi}\, d\Gamma \qquad \otimes$$

$$\underset{\uparrow}{=} \alpha \int_a^b |\gamma'(s)|\,ds + \beta \int_a^b \phi(\gamma(s))\,|\gamma'(s)|\,ds$$

Arclength Parametrization $\gamma : [a,b] \to \mathbb{R}^2$

Compute Variational derivatives $\left(\gamma(t) \to \gamma(t) + \varepsilon \bar\gamma(t)\right)$

$$\frac{d}{d\varepsilon} \int_a^b |\gamma'(s) + \varepsilon \bar\gamma'(s)|\,ds \Big|_{\varepsilon=0} = \int_a^b \frac{1}{|\gamma'(s)+\varepsilon\bar\gamma'(s)|}\left(\bar\gamma'(s)\cdot\gamma'(s) + \varepsilon|\bar\gamma'(s)|^2\right)ds\Big|_{\varepsilon=0}$$

$$= \int_a^b \frac{1}{|\gamma'(s)|}\,\bar\gamma'(s)\cdot\gamma'(s)\,ds$$

$$\underset{\text{Closed Curve}}{=} -\int_a^b \underbrace{\left(\frac{\gamma'(s)}{|\gamma'(s)|}\right)'}_{\substack{=T'(s)\\ =\kappa(s)N(s)}} \cdot \bar\gamma(s)\,ds$$

Variational derivative is defined via

$$\int_a^b (\nabla E_f(\gamma))(s)\cdot\bar\gamma(s)\,ds = \frac{d}{d\varepsilon} E_f(\gamma+\varepsilon\bar\gamma)\Big|_{\varepsilon=0}$$

$$\frac{d}{d\varepsilon} \int_a^b \phi(\gamma(s)+\varepsilon\bar\gamma(s))\,|\gamma'(s)+\varepsilon\bar\gamma'(s)|\,ds\Big|_{\varepsilon=0}$$

$$= \int_a^b \left(\nabla\phi(\gamma(s)+\varepsilon\bar\gamma(s))\cdot\bar\gamma(s)\right)|\gamma'(s)+\varepsilon\bar\gamma'(s)|\,ds\Big|_{\varepsilon=0}$$

$$+ \int_a^b \phi(\gamma(s)+\varepsilon\bar\gamma(s))\,\frac{1}{|\gamma'(s)+\varepsilon\bar\gamma'(s)|}\left(\bar\gamma'(s)\cdot\gamma'(s) + 2\varepsilon|\bar\gamma'(s)|^2\right)ds\Big|_{\varepsilon=0}$$

$$= \int_a^b \left(\nabla\phi(\gamma(s))\cdot\bar\gamma(s)\right)\underbrace{|\gamma'(s)|}_{\substack{\text{Since arclength param.}\\ =1}}\,ds - \int_a^b \kappa(s)N(s)\cdot\bar\gamma(s)\,\phi(\gamma(s))\,ds - \int_a^b \left(\nabla\phi(\gamma(s))\cdot\gamma'(s)\right)\underbrace{\frac{\gamma'(s)}{|\gamma'(s)|}}_{=T}\cdot\bar\gamma(s)\,ds$$

$$= \int_a^b \left(\frac{\partial}{\partial T}\phi(\gamma(s))\,T(s) + \frac{\partial}{\partial N}\phi(\gamma(s))\,N(s)\right)\cdot\bar\gamma(s)\,ds - \int_a^b \kappa(s)N(s)\cdot\bar\gamma(s)\,ds - \int_a^b \left(\frac{\partial}{\partial T}\phi(\gamma(s))\,T(s)\right)\cdot\bar\gamma(s)\,ds$$

Gradient descent: evolve (edge) curve over time via

$$\frac{d}{dt}\gamma(t;s) = -\nabla E_F(\underset{\substack{\uparrow \\ \text{arclength parametrization} \\ \text{of } \Gamma}}{\gamma_s}) = \overbrace{\left(\alpha + \beta\, \phi(\gamma(s))\right) \kappa(s)\, N(s)}^{\text{field modulated mean curvature flow}} - \beta \left(\frac{\partial}{\partial N}\phi(\gamma(s))\right) N(s)$$

Solution of this PDE should converge to (local) minimum of E_F as $t \to \infty$.