Numerical Example:

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Energy Functional of original paper by Kass, Witkin, Terzopoulos:

$$E(\gamma) = \frac{1}{2}\int_0^1 \alpha|\gamma'(s)|^2 + \beta|\gamma''(s)|^2\, ds - \mu \int_0^1 |(\nabla F)(\gamma(s))|\, ds$$

$$\nabla E(\gamma) = -\alpha\gamma''(s) + \beta\gamma''''(s) \underbrace{- \mu\Big(\nabla |\nabla F|\Big)(\gamma(s))}_{= F_{ext}(\gamma(s))}$$

To solve

$$\frac{\partial}{\partial t}\gamma(t,s) = (-\nabla E)(\gamma(t,s)) \qquad (*)$$

$$\gamma''(t,s_i) \approx \gamma(t,s_{i-1}) - 2\gamma(t,s_i) + \gamma(t,s_{i+1})$$

$$\gamma''''(t,s_i) \approx \gamma(t,s_{i-2}) - 4\gamma(t,s_{i-1}) + 6\gamma(t,s_i) - 4\gamma(t,s_{i+1}) + \gamma(t,s_{i+2})$$

$\left(\gamma(t,s_i)\right)_{i=1,\dots,N}$ is a discretization of the curve $\gamma(t,\cdot) = \begin{pmatrix} x(t,\cdot) \\ y(t,\cdot) \end{pmatrix}$

$$\frac{\partial}{\partial t}x(t,s_i) = A \underbrace{\begin{pmatrix} x(t,s_1) \\ \vdots \\ x(t,s_N) \end{pmatrix}}_{x(t,\cdot)} + F^x_{ext}(\gamma(t,s_i))$$

(same for $y(t,s_i)$ with $F_{ext} = \begin{pmatrix} F^x_{ext} \\ F^y_{ext} \end{pmatrix}$)

with

$$A = \begin{pmatrix}
-2\alpha-6\beta & \alpha+4\beta & -\beta & 0 & 0 & \cdots & 0 & -\beta & \alpha+4\beta \\
\alpha+4\beta & -2\alpha-6\beta & \alpha+4\beta & -\beta & 0 & \cdots & 0 & 0 & -\beta \\
-\beta & \alpha+4\beta & -2\alpha-6\beta & \alpha+4\beta & -\beta & 0 \cdots & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & & & & & & & & 0 \\
0 & & & & & & & & \\
-\beta & 0 & & & & -\beta & \alpha+4\beta & -2\alpha-6\beta & \alpha+4\beta \\
\alpha+4\beta & -\beta & 0 & \cdots & 0 & 0 & -\beta & \alpha+4\beta & -2\alpha-6\beta
\end{pmatrix}$$

Diskretization of $\frac{\partial}{\partial t}\gamma(t,s)$

$$\frac{\partial}{\partial t}\gamma(t_i,s) \approx \frac{\gamma(t_i,s)-\gamma(t_{i-1},s)}{\Delta t}, \qquad \text{where } t_i = t_{i-1}\,\Delta t$$

We assume that $F_{ext}(t,s)$ does not change much from t_{i-1} to t_i, then we can write $\otimes$ in a numerically feasible form

$$\frac{X(t_i,\cdot)-X(t_{i-1},\cdot)}{\Delta t} = A\,x(t_i,\cdot) + F^{x}_{ext}(t_{i-1},\cdot)$$

$$\Rightarrow X(t_i,\cdot) = X(t_{i-1},\cdot) + (\Delta t)A\,x(t_i,\cdot) + (\Delta t)F^{x}_{ext}(t_{i-1},\cdot)$$

$$\Rightarrow X(t_i,\cdot) = \left(id-(\Delta t)A\right)^{-1}\left(X(t_{i-1},\cdot) + (\Delta t)F^{x}_{ext}(t_{i-1},\cdot)\right)$$