Optical Flow on Sphere

Expand $f: \mathbb{R}\times S \to \mathbb{R}$ to a function $\bar{f}: \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$. Then

$$\nabla_x \bar{f}(t,x) = \nabla_S \bar{f}(t,x) + \left(\nabla_x \bar{f}(t,x)\cdot N(x)\right) N(x)$$

$$\Rightarrow \nabla_S f(t,x) := \nabla_S \bar{f}(t,x), \quad x\in S$$

Time derivative

$$\frac{d}{dt}\bar{f}(t,x(t)) = \nabla_x \bar{f}(t,x(t))\cdot \frac{d}{dt}x(t) + \frac{\partial}{\partial t}\bar{f}(t,x(t)) \qquad (\ast)$$

$$\frac{d}{dt}\bar{f}(t,x(t)) = \frac{d}{dt}f(t,x(t)), \quad x(t)\in S$$

$$\nabla_x \bar{f}(t,x(t))\cdot \frac{d}{dt}x(t) + \frac{\partial}{\partial t}\bar{f}(t,x(t)) = \nabla_S f(t,x(t))\cdot\frac{d}{dt}x(t) + \frac{\partial}{\partial t}f(t,x(t))$$

The minimizing functional

Let $\{Y_n : n\in\mathbb{N}\}$ be a complete system of orthonormal vectorial basis functions in $L^2(S,\mathbb{R}^3)$, then the following Fourier expansion holds true:

$$u(t,x) = \sum_{n=1}^{N} \hat{u}(n)\, Y_n(x)$$

Then the Functional $\mathcal{F}$ reads (dropping time regularization):

$$\mathcal{F}(u) = \int_S \left(\sum_{n=1}^{N}\hat{u}(n)\,\nabla_S f(t,x)\cdot Y_n(x) + \frac{\partial}{\partial t}f(t,x)\right)^2 dS(x) + \sum_{n=1}^{N}\left(\hat{u}(n)\right)^2 w_n$$

For Minimization differentiate $\mathcal{F}$ with respect to $\hat{u}(n)$:

$$\frac{\partial}{\partial\hat{u}(m)}\mathcal{F}(u) = 2\sum_{n=1}^{N}\left(\hat{u}(n)\int_S \left(\nabla_S f(t,x)\cdot Y_n(x)\right)\left(\nabla_S f(x)\cdot Y_m(x)\right) dS(x)\right) + 2\int_S\left(\frac{\partial}{\partial t}f(t,x)\right)\nabla_S f(t,x)\cdot Y_m(x)\, dS(x)$$

$$+\, 2\, w_m\,\hat{u}(m)$$

The Minimizer u must satisfy $\frac{\partial}{\partial \hat{u}(m)} \mathcal{F}(u) = 0$:

$$\sum_{h=1}^{N} \hat{u}(h) \underbrace{\int_S \left(\nabla_S f(t,x)\cdot y_n(x)\right)\left(\nabla_S f(t,x)\cdot y_m(x)\right) dS(x)}_{=A_{nm}} + \underbrace{w_m}_{=D_{mm}} \hat{u}(m) = \underbrace{-\int_S \left(\frac{\partial}{\partial t} f(t,x)\right) \nabla_S f(t,x)\cdot y_m(x)\, dS(x)}_{b_m}$$

$\Leftrightarrow \quad (A + D)\,\hat{u} = b \quad$ with $\quad \hat{u} = (\hat{u}(1), \dots, \hat{u}(N))^T$

Optical Flow on evolving manifolds

Regard equation ⊛:

Since $\frac{d}{dt}x(t) \overset{= u(x(t))}{}$ does not have to be tangential to S_t, we split it into a tangential and a normal part

$u = u^{tan} + u^{nor}$

Then ⊛ turns into

$$\frac{d}{dt}\bar{f}(t, x(t)) = \nabla_{S_t} \bar{f}(t, x(t)) \cdot u^{tan}(x(t)) + \underbrace{\nabla_x \bar{f}(t, x(t)) \cdot u^{nor}(x(t)) + \frac{\partial}{\partial t}\bar{f}(t, x(t))}_{=: \frac{d^{nor}}{dt}\bar{f}(t, x(t))}, \quad x(t) \in S_t$$