Nonlinear Diffusion

$$\frac{\partial}{\partial t} g(t,x,y) = \nabla \cdot \left(D(|\nabla g(t,x,y)|^2)\, \nabla g(t,x,y) \right)$$

$\frac{\nabla g}{|\nabla g|}$

$g(x) = M$

E.g. $D(|\nabla g|^2) = e^{-\frac{|\nabla g|^2}{2\sigma^2}}$, $D(|\nabla g|^2) = \frac{1}{1+\sigma |\nabla g|^2}$

$$\nabla \cdot \left(D(|\nabla g|^2) \nabla g \right) = D(|\nabla g|^2)\, \Delta g + \left(\nabla D(|\nabla g|^2) \right) \cdot \nabla g$$

$$= D(|\nabla g|^2) \Delta g + 2 D'(|\nabla g|^2) (\nabla g)^T \mathrm{Hess}_g \nabla g$$

$$= D(|\nabla g|^2) \Delta g + 2|\nabla g|^2 D'(|\nabla g|^2) \overbrace{\underbrace{\left(\frac{\nabla g}{|\nabla g|}\right)}_{\text{normal vectors}}^T \mathrm{Hess}_g \frac{\nabla g}{|\nabla g|}}^{\Delta_{normal}}$$

normal vectors in x at level set boundary $\{x \in \mathbb{R}^2 \mid g(x) \leq M\}$ for $M = g(x)$

$$= \underbrace{D(|\nabla g|^2)}_{(**)} \Delta_{tan} g + \underbrace{\left(2|\nabla g|^2 D'(|\nabla g|^2) + D(|\nabla g|^2) \right)}_{(*)} \Delta_{normal} g$$

$\Delta = \Delta_{tan} + \Delta_{normal}$

if $|\nabla g| \gg 1$, $(**) > (*)$ (i.e. tangential diffusion) dominates

if $|\nabla g| \ll 1$, $(**) \approx (*)$ (approx. isotropic diffusion)