Visualization and Imaging

Exercise Sheet 1

Exercise 1:

Implement the marching squares algorithm in MATLAB such that it takes a 2-D image f (i.e., a matrix $I_f \in \mathbb{R}_+^{N\times M}$) and a threshold K as input and eventually plots the resulting isocurve corresponding to the threshold K.

Exercise 2:

Regard the functional

$$\mathcal{F}(u) = \int_{\mathbb{S}^2} \left(\nabla_{\mathbb{S}^2} f(t,x) \cdot u(x) + \frac{\partial}{\partial t} f(t,x) \right)^2 dS(x) + \|u\|^2_{H_1(\mathbb{S}^2)},$$

which we have derived for optical flow on the sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a dense orthonormal function system in $L^2(\mathbb{S}^2, \mathbb{R}^3)$, i.e.,

$$\int_{\mathbb{S}^2} y_n(x) \cdot y_m(x) dS(x) = \begin{cases} 0, & n \neq m, \\ 1, & n = m, \end{cases}$$

and

$$u(x) = \sum_{n=0}^{\infty} \alpha_n y_n(x), \tag{1}$$

for adequate coefficients $\alpha_n \in \mathbb{R}$. Furthermore, we may assume that the squared norm $\|u\|^2_{H_1(\mathbb{S}^2)}$ can be expressed as

$$\|u\|^2_{H_1(\mathbb{S}^2)} = \sum_{n=0}^{\infty} \alpha_n^2 w_n,$$

for some known weights $w_n \in \mathbb{R}_+$. At last, let us assume that u from (1) can be expressed by a finite sum

$$u(x) = \sum_{n=0}^{N} \alpha_n y_n(x), \tag{2}$$

for some predefined maximum $N \in \mathbb{N}$. In other words, the function u is defined by the coefficients α_n, $n = 0, \ldots, N$.

Under the conditions above, show that the minimizer u of the functional $\mathcal{F}$ can be determined by solving the linear equation

$$(A + D)\alpha = b,$$

where $\alpha = (\alpha_0, \ldots, \alpha_N)^T$ and

$$\begin{aligned}
A &= (A_{n,m})_{n,m=0,\ldots,N}, \qquad & A_{n,m} &= \int_{\mathbb{S}^2} \left(\nabla_{\mathbb{S}^2} f(t,x) \cdot y_n(x)\right) \left(\nabla_{\mathbb{S}^2} f(t,x) \cdot y_m(x)\right) dS(x), \\
D &= (D_{n,m})_{n,m=0,\ldots,N}, \qquad & D_{n,m} &= \begin{cases} 0, & n \neq m, \\ w_n, & n = m, \end{cases} \\
b &= (b_0, \ldots, b_N)^T, \qquad & b_n &= -\int_{\mathbb{S}^2} \left(\nabla_{\mathbb{S}^2} f(t,x) \cdot y_n(x)\right) \frac{\partial}{\partial t} f(t,x) dS(x)
\end{aligned}$$

Exercise 3:

Regard the functional

$$\mathcal{F}(g) = \alpha \int_{\Omega} |\nabla g(x)|^2 dx + \mu \int_{\Omega} (g(x) - f(x))^2 dx$$

for some possibly noisy image $f \in L^2(\Omega, \mathbb{R})$ and a (smoothed) filtered version $g \in L^2(\Omega, \mathbb{R})$ (additionally assuming that $\nabla g \in L^2(\Omega, \mathbb{R}^3)$), where $\Omega = [0,1]^2 \subset \mathbb{R}^2$. Show that minimizing the functional $\mathcal{F}$ is equivalent to solving the differential equation

$$\begin{aligned}
-\alpha \Delta g(x) + \mu g(x) = \mu f(x), &\qquad x \in \Omega \\
\frac{\partial}{\partial \nu} g(x) = 0, &\qquad x \in \partial\Omega,
\end{aligned}$$

where $\frac{\partial}{\partial \nu} g$ denotes the normal derivative of g on the boundary $\partial\Omega$ of Ω. (Hint: Compute the first variation and use Green's identities)

Remark: Using a steepest descent method, one could also derive the reaction diffusion equation

$$\begin{aligned}
\frac{\partial}{\partial t} g(t,x) &= \alpha \Delta g(t,x) + \mu(f(x) - g(t,x)), \qquad x \in \Omega, t > 0, \\
\frac{\partial}{\partial \nu} g(t,x) &= 0, \qquad x \in \partial\Omega, t > 0,
\end{aligned}$$

for adequate initial conditions $g(0,x)$. Compare this to diffusion filtering described in the lecture.