Integral Invariants
and
Shape Matching

- I propose to hold this seminar in English since my knowledge of German is insufficient.
  I am at your disposal for comments: steven.verpoort@gmail.com
  I will make this document available in case of interest.

- This seminar is a discussion of the paper
  [1] „Integral Invariants and Shape Matching"
  by S. Manay, D. Cremers, B-W. Hong, A. Yezzi, S. Soatto
  published as a chapter in: H. Krim and A. Yezzi (Eds.),
  „Statistics and analysis of Shapes" (Birkhäuser 2006)

§1 Introduction

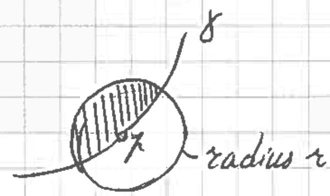
The shape of a curve in $\mathbb{R}^2$ is described by its <u>curvature</u> as a function of arc-length. This function $\kappa = \kappa(s)$ uniquely characterises the curve up to a Euclidean transformation, hence the name „<u>natural equation</u>".

This curvature $\kappa(s)$ is a so-called <u>differential invariant</u>, i.e., the $\kappa(s)$ which is associated to the point $\gamma(s)$ of the curve γ depends only on $\gamma(s)$ and a finite number of derivatives of γ in s, and it is invariant w.r.t. Euclidean transformations.

However in applications ~~sh~~ such as computer vision the curve γ suffers from noise and hence the inverse question arises how κ changes if γ is perturbed. The curvature κ is highly sensitive to noise because it involves derivatives.

The traditional remedy to overcome this high ~~seamd~~ sensivity to noise is to make the curve smoother in a first stage.
In the article under discussion is proposed to study <u>integral invariants</u>, i.e., invariants obtained by the process of integration and which are therefore less sensitive to noise.

A typical example is the <u>local area invariant</u> which, for fixed r, assigns to a point p on the curve the area of the shaded domain (Fig.)



In the sequel of this lecture I will follow approximately the outline of the article:

- §2. Curvature and Differential Invariants
- §3. Integral Invariants
- §4. Relation between Curvature and Integral Invariants
- §5. Shape Matching and Shape Distance
- §6. Multiscale Invariants
- §7. A few remarks about implementation & experimental results.

(Implementation will also be discussed normally by Mr. R. Rexhaj.)

§2. Curvature and Differential Invariants.

Before discussing alternatives and modifications it seems appropriate to briefly recall the definition of curvature (of a curve in the Euclidean plane).

Such a curve is a mapping assigning a point in $\mathbb{R}^2$ to each parameter value t,

$$\gamma : \mathbb{R} \to \mathbb{R}^2 : \quad t \mapsto \gamma(t)$$

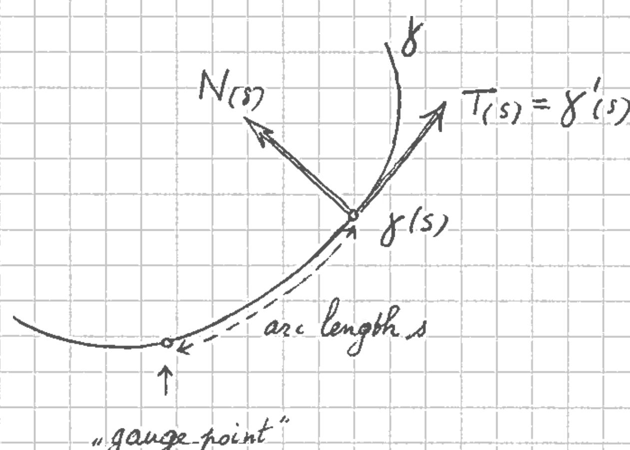
First the arc-length parameter s is introduced, $s = s(t)$, ~~by~~

To every point on the curve a „t-value" as well as an „s-value" corresponds, and (with a slight abuse in notation) we write both $\gamma(t)$ and $\gamma(s)$.

The chain rule holds:
$$\frac{\partial \gamma}{\partial t} = \frac{\partial s}{\partial t} \cdot \frac{\partial \gamma}{\partial s}$$

and s is an arc-length parameter if $\left\| \frac{\partial \gamma}{\partial s} \right\| = 1$, which gives the condition
$$\frac{\partial s}{\partial t} = \pm \left\| \frac{\partial \gamma}{\partial t} \right\|$$
on the function $s = s(t)$.

Then mostly one denotes T for γ', where $' = \frac{\partial}{\partial s}$, and N for the vectorfield obtained from T by a rotation over an angle of $\frac{\pi}{2}$.



The curvature κ is then defined as the signed rate of twist of this frame $\{T, N\}$ as a function of s.

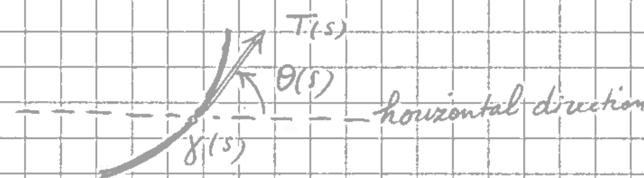
From $1 = \langle T, T \rangle$ follows $0 = \langle T', T \rangle$ hence $T' \propto N$

We introduce κ as the factor of proportionality

$$\boxed{T' = \kappa N}$$

We mention three formulae for κ.

- $\boxed{\kappa = \frac{1}{R}}$ radius of osculating circle
- If, for $\theta : \mathbb{R} \to \mathbb{R} : s \mapsto \theta(s)$, we have $T(s) = \cancel{\begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix}} \begin{pmatrix} \sin\theta(s) \\ \cos\theta(s) \end{pmatrix}$
  then
  $\boxed{\kappa = \frac{\partial \theta}{\partial s}}$ (Euler)



- If $\gamma(t) = (x(t), y(t))$, then

$$\boxed{\kappa(t) = \frac{x'(t)\, y''(t) - x''(t)\, y'(t)}{\left((x'(t))^2 + (y'(t))^2 \right)^{3/2}}}$$ (Newton)

We recall the <u>fundamental theorem</u>:

- Uniqueness: If α and β are arc-wise parametrised curves and $\kappa_\alpha(s) = \kappa_\beta(s)$ then $\exists\, \varphi \in \mathrm{Euc}$ [the collection of Euclidean transformations] for which
  $$\alpha = \varphi \circ \beta .$$
- Existence: For every function $\kappa = \kappa(s)$ a curve can be found which realises κ as curvature function.

As is expressed by Newton, κ can be calculated from γ and a finite number of its derivatives, and this w.r.t. an arbitrary parametrisation (not only arclength-parametrisation). This leads to the following definition.

Def. A <u>differential invariant</u> for curves $\gamma(t)=(x(t),y(t))$ in the plane is a function

$$F(\gamma(t),\gamma'(t),\gamma''(t),\dots,\gamma^{(n)}(t))$$

which satisfies the two properties

($1^{\underline{o}}$) Invariance w.r.t. Euclidean motions, i.e., if $Euc=\{\text{Euclidean motions}\}$ we require $\forall \varphi\in Euc$, $\forall$ curve γ, $\forall$ parameter value t

$$F(\gamma(t),\gamma'(t),\dots,\gamma^{(n)}(t)) = F((\varphi\circ\gamma)(t),(\varphi\circ\gamma)'(t),\dots(\varphi\circ\gamma)^{(n)}(t))$$

($2^{\underline{o}}$) Invariance wrt reparametrisation, i.e.
$\forall$ curve γ and $\forall$ reparametrisation $\sigma:\mathbb{R}\to\mathbb{R}$,
$\forall$ ~~parameter value~~ t

$$F(\gamma,\gamma',\dots,\gamma^{(n)})\circ\sigma = F((\gamma\circ\sigma),(\gamma\circ\sigma)',\dots,(\gamma\circ\sigma)^{(n)})$$

(Remark: More precisely this is the definition of a Euclidean differential invariant and one could also consider the group of equi-affine / affine / projective / Möbius / ⋮ $\Big\}$ differential invariants, ∄

(Remark: the differential invariant depends on the curve γ and on a point $\gamma(t)$ on it.)

§3. Integral Invariants.

In the article the notion of integral invariant is proposed, which

- is obtained by integration (instead of differentiation)
- assigns a number to a curve γ and a point p not necessarily lying on γ.

Def. <u>An integral invariant</u> for curves $\gamma(t) = (x(t), y(t))$ in the plane is a function

$$I : \{\text{curves}\} \times \mathbb{R}^2 \to \mathbb{R} : \quad (\gamma, p) \mapsto I_\gamma(p) \tag{7-1}$$

which

- has been constructed from a function $h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

  as $$I_\gamma(p) = \int_{\bar{\gamma}} h(p, x)\, d\mu_{(x)} \tag{7-2}$$

  or $$I_\gamma(p) = \int_\gamma h(p, \gamma(t))\, \left|\frac{d\gamma}{dt}\right| dt \tag{7-3}$$

- The function h is such, that the corresponding $I_\gamma(p)$ is effectively an invariant:

  $$I_\gamma(p) = I_{\varphi \circ \gamma}(\varphi(p))$$

  for all $\cancel{\gamma}$ $p \in \mathbb{R}^2$ and every curve γ, and all $\varphi \in \mathrm{Euc}$

(Remark: In eq. (7-2) we integrated over the bounded domain $\bar{\gamma}$ with boundary γ, and $d\mu_{(x)}$ is the area element of $\mathbb{R}^2$. In eq (7-1) we took a path-integral along the curve, and $\left|\frac{d\gamma}{dt}\right| dt$ stands for integration w.r.t. arc-length.)

(Remark: If ~~h depend~~ $h(x, y)$ depends only on $|x-y|$ then the second requirement is fulfilled

Example: the local area invariant for a certain r. (Cf. p.)

Example: <u>the distance integral invariant</u>

$$I_\gamma(p) = \int_\gamma |p - \gamma(s)|\, ds$$

Corresponds to the choice

~~$h(x, y) = |x - y|$~~

$h(x, y) = 1$ (if $|x-y| \leq$

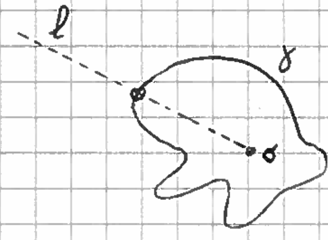
$\quad = 0$ (if $|x-y| >$

~~Example:~~

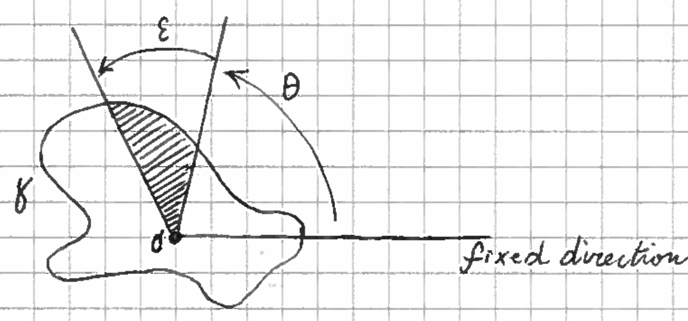
There is another invariant which we should mention but which ~~depends~~ is only invariant w.r.t. „centro-Euclidean transformations":
Fix a point σ (the origin, the centre) in $\mathbb{R}^2$ and set

$$\begin{aligned} cEuc &= \{ \text{centro-Euclidean transformations} \} \\ &= \{ \text{Euclidean transformations for which } \varphi(\sigma) = \sigma \}. \end{aligned}$$

A curve γ is said to be starshaped w.r.t. the point σ if every half-line ℓ emanating from σ meets γ once.



Further let $\varepsilon > 0$ be given. The cone area integral invariant with aperture ε assigns to the angle Θ the area of the shaded domain (below).



→ The important point is that integral invariants are more robust w.r.t. perturbation of the curves, as shown in Fig 2 of the article.

→ The question remains open for which integral invariants we really do have a fundamental theorem of existence and uniqueness.
One of the rare results in ~~this~~ this direction is obtained by Fidler/Grasmair/Scherzer [2]:

They showed that for almost every ε, If the local cone area integral invariant of a starshaped curve γ is ~~given~~ known at a function of θ, then γ is uniquely determined.

→ Let now $I_r(s)$ be the local area invariant of a curve γ, with r the disk radius:

$$I_r(s) = \text{area of } \bar{\gamma} \text{ contained inside a disk of radius } r \text{ and centre } \gamma(s).$$

If, for all r and s, $I_r(s)$ is known, then $\kappa(s)$ is known (as follows from the result of the next §) and hence the curve is uniquely determined, up to motions.

But this is not a „good" theorem because this merely says a curve is known if a function of 2 variables is known (whereas $\kappa(s)$ depends only of 1 variable)

§4. Relation between the local area invariant $I_r(s)$ and curvature $\kappa(s)$

For a curve γ, there holds

$$I_r(s) = \frac{\pi}{2} r^2 - \frac{\kappa(s)}{3} r^3 + \mathcal{O}(r^4) . \qquad (9\text{-}1)$$

(The equation given in [1], § 3.1, is not correct. See [3], eq. (11).)

This implies

$$\kappa(s) = \lim_{r \to 0} \frac{\frac{\pi}{2} - I_r(s)/r^2}{r} . \qquad (9\text{-}2)$$

For completeness the proof of (9-1) can be found on the next pages.

Furthermore the curve can be parametrised as the graph of f,

$$x \mapsto (x, f(x)),$$

hence Newton's formula becomes

$$\kappa = \frac{''(x)}{1+(f'(x)^2))^{3/2}}$$

and hence

$$\frac{\partial}{\partial s}\kappa = \frac{\partial x}{\partial s}\frac{\partial \kappa}{\partial x} = \frac{1}{\sqrt{1+(f'(x))^2}} \cdot \left(\frac{f'''(x)}{(1+(f(x))^2)^{3/2}} - \frac{3}{2}\,\frac{2 f'(x)\,(f''(x))^2}{(1+(f'(x))^2)^{5/2}} \right).$$

Evalutaion in the origin gives

$$\begin{cases} \kappa(0) = f''(0) \\ \frac{\partial \kappa}{\partial s}(0) = f'''(0) \end{cases}$$

and hence

$$\gamma \longleftrightarrow y = \frac{\kappa(0)}{2}x^2 + \frac{\frac{\partial\kappa}{\partial s}(0)}{6}x^3 + \mathcal{O}(x^4)$$

Now we search for the two points of intersection of γ and this circle, which are themselves dependent on r.

Their co-ordinates (x, y) should satisfy

$$r = x^2 + y^2 = x^2 + \frac{\kappa(0)^2}{4}x^4 + \frac{\kappa(0)\frac{\partial\kappa}{\partial s}(0)}{6}x^5 + \mathcal{O}(x^6) \qquad (11\text{-}*)$$

and we should solve this in x (as function of r). Write the solution as $x = A\cdot r + B\cdot r^2 + C\cdot r^3 + \mathcal{O}(r^4)$ and substitute this in the right hand side above, then we should obtain $r^2 = r^2 + \mathcal{O}(x^6)$.

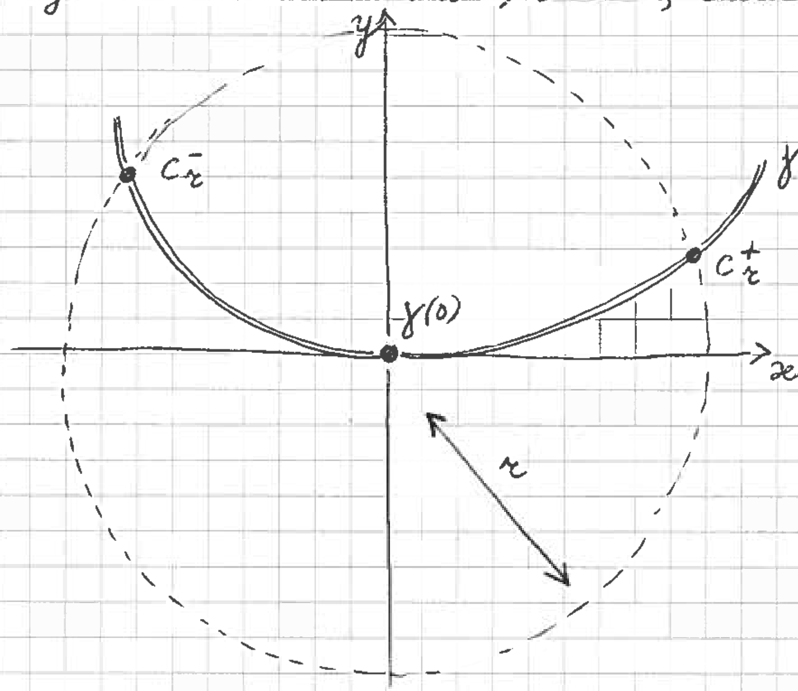
This yields the unknown coefficients A, B, C, namely:

$$\begin{cases} (\text{for } C_r^+) \qquad x^+(r) = r - \frac{\kappa(0)^2}{8}r^3 + \mathcal{O}(r^4) \\ (\text{for } C_r^-) \qquad x^-(r) = -r + \frac{\kappa(0)^2}{8}r^3 + \mathcal{O}(r^4) \end{cases} \qquad (11-2)$$

<u>Proof of (9-1)</u>. For a fixed curve γ, parametrised by arc-length s let $I_r(s)$ be the local area invariant, i.e., area of the disk segment; let $L_r(s)$ be the length of the circular segment (cf. fig below)

Obviously $$I_r(s) = \int_0^r L_{\underline{r}}(s)\, d\underline{r}.$$

It is not a restriction to prove (9-1) only at $s=0$. Let us introduce rectangular co-ordinates as below, it means, choose $T(0)$ and $N(0)$ as axes.



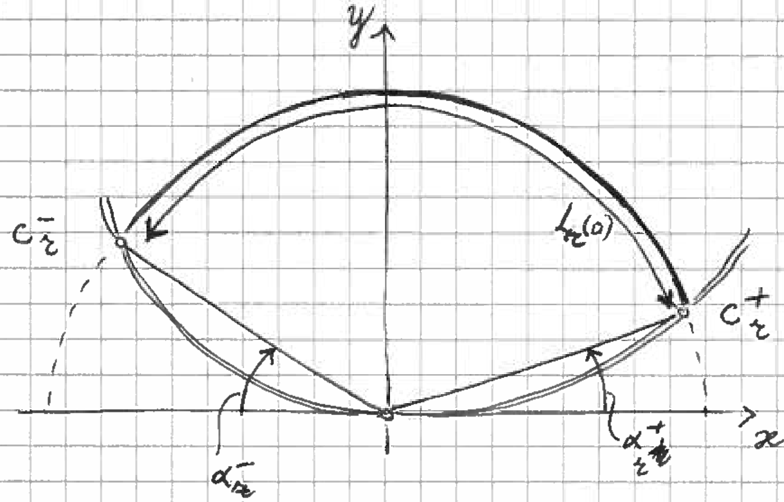
Around the point $\gamma(0)$, the curve admits the representation

$$\gamma \leftrightarrow y = f(x)$$

where obviously

$f(0)=0$ because $\gamma(0)$ is the origin

$f'(0)=0$ because we chose the x-axis tangent to the ~~surface~~ curve.



Introduce the angles α_r^+, α_r^- as on the figure.

From (11-2) and the formula

$$\begin{cases} \alpha_r^+ = \arccos\left(\frac{x_r^+}{r}\right) \\ \alpha_r^- = \arccos\left(\frac{-x_r^-}{r}\right) \end{cases}$$

we can calculate α_r^+, α_r^- as function of r.

Finally, $L_r(0) = (\pi - \alpha_r^- - \alpha_r^+) \cdot r$

$$= \pi \cdot r - K_{(0)} r^2 + \mathcal{O}(r^3)$$

$$I_r(\gamma(0)) = \frac{\pi}{2} r^2 - \frac{K(0)}{3} r^3 + \mathcal{O}(r^4)$$

(to be continued)