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LOCAL UNIQUENESS OF THE CIRCULAR INTEGRAL INVARIANT

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Abstract. This article is concerned with the representation of curves by means of integral invariants. In contrast to the classical differential invariants they have the advantage of being less sensitive with respect to noise. The integral invariant most common in use is the circular integral invariant. A major drawback of this curve descriptor, however, is the absence of any uniqueness result for this representation. This article serves as a contribution towards closing this gap by showing that the circular integral invariant is injective in a neighbourhood of the circle. In addition, we provide a stability estimate valid on this neighbourhood. The proof is an application of Riesz–Schauder theory and the implicit function theorem in a Banach space setting.

1. Introduction.

In many applications one faces the challenge to model objects, or parts of objects, in a mathematical framework. As an example, one important task is to extract an object from a given data set and manipulate it in a post-processing step in order to obtain further information. Typical applications include medical imaging, object tracking in a sequence of images, but also object recognition, where the post-processing step consists of the comparison of the extracted object with a database of reference objects. Similarly, such a comparison can be necessary in medical imaging in order to distinguish between healthy and diseased organs. To that end, however, one has to be able to decide whether two given objects are similar or not. This requires a representation of the objects that makes the application of standard similarity measures possible.

Finding a suitable representation of the object of interest, depending on the type of application, is crucial as a first step. For simplicity, it is often assumed that the

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object is a simply connected bounded domain, allowing for the identification of the domain with its boundary. From a mathematical point of view, this assumption reduces the complexity of the representation. In addition, there exists a larger number of descriptions of boundaries than of domains, and, consequently, more mathematical tools to analyze the geometry of the underlying objects.

In 2D a common approach is to encode the contour of an object by the curvature function of its boundary curve. This approach has, for instance, been used in [8], where the authors set up a shape space of planar curves, where the shapes are implicitly encoded by the curvature function. The main advantage of using a differential invariant — the most prominent representative being the curvature — to represent an object is the well investigated mathematical framework of this type of invariants (see [1, 10, 13]).

Since all kinds of differential invariants are based on derivatives, they suffer from the shortcoming of being sensitive with respect to small perturbations. To bypass this shortcoming Manay et al. [11] proposed to use integral invariants instead of their differential counterparts (see also [5, 7, 16]). Integral invariants have similar invariance properties as differential invariants, but have proven to be considerably more robust with respect to noise. Their theory, however, is not that well investigated as opposed to the theory of their differential counterparts.

Beside the classical approach of differential invariants and the novel approach of integral invariants, there exist several other concepts for encoding an object. For instance, in [3] the authors use the zero level set of a harmonic function, which is uniquely determined by prescribing two functions on the boundary of an annulus, to encode the boundary of a 2D object (see [4] for a generalization to compact surfaces in 3D). A similar encoding of the object by a function is given in the article of Sharon and Mumford [15]. Here, the authors first map the 2D object, which is supposed to be a smooth and simply closed curve, to the interior of the unit disc in the complex plane via the Riemann mapping theorem. This conformal mapping is composed with a second one, generated out of the exterior of the original object, and the composition is restricted to the boundary of the unit disc. Thus, the final mapping, which the authors call the fingerprint of the object, is a diffeomorphism from the unit circle onto itself.

One of the challenges in object encoding is the question of uniqueness of the encoding. More precisely, in many applications, e.g. object matching, the correspondence between the object and its encoding should be one-to-one. Thus, a thorough investigation of the operator that maps an object to its encoding is needed. In case of the encoding by a harmonic or conformal mapping — if possible — uniqueness is well known. Also for the encoding of an arc length parameterized curve by its curvature function, it is known that one obtains a one-to-one correspondence between the curve and its encoding (up to rigid body motions). One even has a complete characterization of the set of functions that arise as curvature functions of a class of sufficiently regular curves (see [2]). For integral invariants the situation is different; the cone area invariant, first introduced in [5], is an injective mapping independent of the space dimension, but its application is limited to star-shaped objects. In contrast, for the circular integral invariant, which is the integral invariant most common in use, there exists no proof for the uniqueness conjecture so far.

This article is a contribution towards this goal: We first prove that the integral invariant is ℓ-times continuously Fréchet differentiable in a neighborhood of the circle, seen as a mapping from $C^{k+\ell+1}$ to C^k, $k \geq 0$. Then we show that the Fréchet differential is injective on some $C^{k+\ell+1}$-neighborhood of the circle, $k \geq 1$, $\ell \geq 1$. The proofs of these results are based on the implicit function theorem on Banach spaces and an application of Riesz–Schauder theory. Using the injectivity result, a Taylor series expansion, and an interpolation inequality for C^m norms, we obtain the injectivity of the integral invariant on a C^{k+6} neighborhood $\mathcal{V}$, $k \geq 1$. More precisely, we show that, in $\mathcal{V}$, the C^k-norm of the difference of the integral invariants of two curves can be estimated from below by their C^k-distance.

2. **Setting.** Let Emb be the space of all continuous embeddings from S^1 to $\mathbb{R}^2$. Then every curve $\gamma \in \mathrm{Emb}$ has a unique interior, denoted by $\mathrm{Int}(\gamma)$. Following [5, 11], this allows us to introduce the circular integral invariant:

Definition 2.1. For given $r > 0$ we define the *circular integral invariant*

$$I_r[\gamma] \colon S^1 \to \mathbb{R}$$

of a curve $\gamma \in \mathrm{Emb}$ as

$$I_r[\gamma](\varphi) := \mathrm{area}\big(B_r\big(\gamma(\varphi)\big) \cap \mathrm{Int}(\gamma)\big) ,$$

where $B_r(p)$ denotes the ball of radius r centered at $p \in \mathbb{R}^2$.

The circular integral invariant behaves well under several group actions:

- I_r is invariant with respect to Euclidean motions: For $A \in \mathrm{SE}[2]$ we have
$$I_r[A \circ \gamma] = I_r[\gamma] .$$
- I_r is equivariant with respect to reparametrizations: For every homeomorphism $\Phi \colon S^1 \to S^1$ we have
$$I_r[\gamma \circ \Phi] = I_r[\gamma] \circ \Phi .$$
- For every scalar $t > 0$ we have
$$I_r[t\gamma] = t^2 I_{r/t}[\gamma] .$$

The observations above suggest to consider the integral invariant on the space $\mathcal{C}$ of all curves modulo Euclidean motions and reparametrizations. Moreover, we assume as an additional smoothness property that the considered curves are of class C^k, $k \geq 1$. Then it makes sense to use the following representation of $\mathcal{C}$, as it avoids working with equivalence classes of curves.

Definition 2.2. Denote by $\mathcal{C}_k \subset \mathrm{Emb}$, $k \geq 1$, the space of all curves $\gamma \in C^k(S^1;\mathbb{R}^2)$ satisfying the following conditions:

- γ has constant speed, i.e., there exists a constant $c_\gamma > 0$ such that $\|\dot\gamma(\varphi)\| = c_\gamma$ for all $\varphi \in S^1$.
- $\gamma(0) = (1,0)$ and $\dot\gamma(0) = (0, c_\gamma)$, where we identify the circle S^1 with the interval $[0, 2\pi)$.
- γ is an embedding, that is, $\gamma(\varphi) \neq \gamma(\psi)$ for all $\varphi \neq \psi$.

For the proof of our main theorem we need the following result from differential geometry concerning the manifold structure of $\mathcal{C}_k$:

Theorem 2.3. *For $k \geq 1$ the space $\mathcal{C}_k$ is a smooth submanifold of the Banach space of all C^k-curves from S^1 to $\mathbb{R}^2$. Its tangent space $T_\gamma\mathcal{C}_k$ at a curve $\gamma \in \mathcal{C}_k$ consists of all C^k-curves σ with*

$$\langle \dot\sigma(\varphi), \dot\gamma(\varphi)\rangle = c \text{ for some } c \in \mathbb{R}, \quad \sigma(0) = (0,0) \text{ and } \langle \dot\sigma(0), \gamma(0)\rangle = 0 .$$

Proof. The proof of the submanifold result is similar to [14, Thm. 2.2]. In our case the situation is less complicated, as we only deal with C^k-curves instead of Sobolev curves. The constant speed parameterization yields the condition

$$2c = \partial_\varepsilon|_0 \langle \dot\gamma(\varphi) + \varepsilon\dot\sigma(\varphi), \dot\gamma(\varphi) + \varepsilon\dot\sigma(\varphi)\rangle = 2\langle \dot\sigma(\varphi), \dot\gamma(\varphi)\rangle\ .$$

The remaining constraints follow directly from the initial conditions. □

Under additional smoothness assumptions on γ we obtain the following characterization of the tangent space $T_\gamma\mathcal{C}_k$.

Lemma 2.4. *Let $k \geq 1$ and $\gamma \in \mathcal{C}_k \cap C^{k+1}(S^1;\mathbb{R}^2)$ with curvature function*

$$\kappa_\gamma(\varphi) := \frac{\langle \dot\gamma(\varphi)^\perp, \ddot\gamma(\varphi)\rangle}{c_\gamma^3}.$$

Then the tangent space $T_\gamma\mathcal{C}_k$ consists of all C^k-curves $\sigma(\varphi) = a(\varphi)\dot\gamma(\varphi)^\perp + b(\varphi)\dot\gamma(\varphi)$ satisfying:

- $\dot b(\varphi) = \dot b(0) + a(\varphi)\kappa_\gamma(\varphi)c_\gamma$.
- $a(0) = b(0) = 0$.
- $\dot a(0) = 0$.

Proof. Theorem 2.3 and the fact that $\langle \dot\gamma(\varphi), \ddot\gamma(\varphi)\rangle = 0$ imply that there exists a constant $c \in \mathbb{R}$ such that

$$\begin{aligned} c = \langle \dot\sigma(\varphi), \dot\gamma(\varphi)\rangle &= \langle \dot\gamma(\varphi), \dot a(\varphi)\dot\gamma(\varphi)^\perp + a(\varphi)\ddot\gamma(\varphi)^\perp + \dot b(\varphi)\dot\gamma(\varphi) + b(\varphi)\ddot\gamma(\varphi)\rangle \\ &= a(\varphi)\langle \dot\gamma(\varphi), \ddot\gamma(\varphi)^\perp\rangle + \dot b(\varphi)c_\gamma^2 = -a(\varphi)c_\gamma^3\kappa_\gamma(\varphi) + \dot b(\varphi)c_\gamma^2\ . \end{aligned}$$

Using the initial conditions for σ, we obtain the initial conditions for a and b and the value of $c = c_\gamma^2\dot b(0)$. □

We are now able to formulate the main result of this article. Here and in the following we denote by $\|\gamma\|_k$ the C^k norm on the space of curves. Similarly, if $F\colon C^k \to C^\ell$ is a bounded linear operator, we denote its operator norm by $\|F\|_{k,\ell}$, and we use the same notation for norms of multi-linear operators.

Theorem 2.5. *The circular integral invariant $I_r\colon \mathcal{C}_{k+\ell+1} \to C^k(S^1;\mathbb{R})$, $k \geq 1$, $\ell \geq 1$ is ℓ-times continuously Fréchet differentiable on a neighborhood $\mathcal{U} \subset \mathcal{C}_{k+\ell+1}$ of the circle of radius $R > r/2$ and its tangential mapping is injective on this neighborhood.*

Moreover there exists a neighborhood $\tilde{\mathcal{U}}$ of the circle with respect to the topology induced by the C^{k+6}-norm and a constant $c > 0$ such that for every γ, $\tilde\gamma \in \tilde{\mathcal{U}}$ the stability estimate

$$\|I_r[\gamma] - I_r[\tilde\gamma]\|_k \geq c\|\gamma - \tilde\gamma\|_k$$

holds. In particular, the mapping I_r is injective on $\mathcal{V}$.

Remark 1. The condition on r to be smaller than $2R$ is necessary, because otherwise the circular integral invariant in each point φ is constant equal to $R^2\pi$, the area of the circle, and the same holds for any sufficiently small deformation of the circle which preserves the area.

3. **Fréchet Differentiability of the Circular Integral Invariant.** In the following, we discuss the differentiability of I_r and derive an analytic formula for I_r and, under certain smoothness assumptions, its derivative I_r' valid in a neighborhood of the circle. As a first step, we recall the following result on the differentiability of the composition mapping. To that end, we need the following definitions of differentiability of mappings on Banach spaces.

Definition 3.1. Let X, Y be Banach spaces, $F\colon X \to Y$, and $\ell > 1$. The mapping F is called ℓ-times *weakly differentiable*, if it is ℓ-times Gâteaux differentiable and its Gâteaux differential $d^\ell F$ is continuous as a mapping

$$d^\ell F\colon X^{\ell+1} \to Y.$$

In contrast, it is called ℓ-times *continuously Fréchet differentiable* or *of class* C^ℓ, if it is ℓ-times Gâteaux differentiable and $d^\ell F$ is continuous as a mapping

$$d^\ell F\colon X \to L^\ell(X, Y).$$

Here $L^\ell(X, Y)$ is the Banach space of ℓ-linear mappings from X^ℓ to Y equipped with the operator norm.

Note that the continuity requirement for weak differentiability is weaker than for continuous Fréchet differentiability, and thus a weakly differentiable mapping need not be Fréchet differentiable of the same order. The following Lemma shows, however, that it is Fréchet differentiable of lower order.

Lemma 3.2. *Let X, Y be Banach spaces and $F\colon X \to Y$ $(\ell+1)$-times weakly differentiable with $\ell \geq 1$. Then F is ℓ-times continuously Fréchet differentiable.*

Proof. Let $x \in X$ and $\varepsilon > 0$.

We have to show that there exists $\delta > 0$ such that for every $y \in X$ with $\|x-y\| < \delta$ the inequality

$$\|d^\ell F(x) - d^\ell F(y)\|_{L^\ell(X,Y)} = \sup_{\substack{z\in X^\ell\\ \|z\|_{X^\ell}=1}} \|d^\ell F(x)(z) - d^\ell F(y)(z)\|_Y < \varepsilon$$

holds.

Using the continuity of $d^{\ell+1}F\colon X^{\ell+2} \to Y$ at the point $(x, 0, 0) \in X \times X \times X^\ell$ and the fact that $d^{\ell+1}F(x, 0, 0) = 0$, we obtain the existence of $\eta > 0$ such that

$$\left\|d^{\ell+1}F(x_1, x_2, \tilde z)\right\| \leq \varepsilon \qquad \text{whenever} \qquad \|x_1 - x\| + \|x_2\| + \|\tilde z\| < 3\eta.$$

Now let $\delta := \min\{\eta^{\ell+1}, 1\}$, let $y \in X$ with $\|y - x\| < \delta$ and $z \in X^\ell$ with $\|z\| \leq 1$. Then we have

$$\begin{aligned}
\|d^\ell F(x)(z) - d^\ell F(y)(z)\|_Y &= \left\|\int_0^1 \partial_t d^\ell F(x + t(y-x), z)\,dt\right\| \\
&= \left\|\int_0^1 d^{\ell+1} F(x + t(y-x), y-x, z)\,dt\right\| \\
&\leq \int_0^1 \left\|d^{\ell+1} F(x + t(y-x), y-x, z)\right\|\,dt \\
&= \int_0^1 \left\|d^{\ell+1} F\Big(x + t(y-x), \frac{y-x}{\eta^\ell}, \eta z\Big)\right\|\,dt < \varepsilon.
\end{aligned}$$

$\square$

Lemma 3.3. *For every $k \geq 0$, $\ell \geq 1$ the composition mapping*

$$\begin{aligned}\mathrm{Comp}\colon C^{k+\ell+1}(S^1;\mathbb{R}^2) \times \mathrm{Diff}^k(S^1) &\to C^k(S^1;\mathbb{R}^2)\,,\\ (f,g) &\mapsto f \circ g\,,\end{aligned}$$

is $(\ell+1)$-times weakly differentiable and therefore ℓ-times continuously Fréchet differentiable.

Proof. The result on weak differentiability has been shown in [12, Section 6.9] (note that the result in [12] has been shown for the space HC^n, which, however, is equivalent to C^n in the case of a compact manifold). The statement concerning the continuous Fréchet differentiability follows from Lemma 3.2. □

Remark 2. In order to simplify the notation, we will sometimes omit the domain and range of the function spaces in expressions like $C^k(S^1;\mathbb{R})$ and write C^k instead, if no confusion is possible.

Remark 3. We will need two different types of derivatives for the formulation of our results: First, Fréchet derivatives in the function space C^k, and, second, derivatives of functions $f \in C^k(S^1)$ with respect to their argument $\varphi \in S^1$. In order to highlight this difference, we use the following notation: For a function $F\colon C^k \to C^j$, we denote by $F'\colon C^k \to L(C^k, C^j)$ its Fréchet derivative. In contrast, if $f \in C^k$, then $\dot{f}$ denotes its derivative in the parameter space.

In order to make the notation less cumbersome, we omit the argument φ in $\gamma(\varphi)$, $\sigma(\varphi)$ and similar expressions if the argument is clear from the context.

Lemma 3.4. *For each $k \geq 0$, $\ell \geq 1$ there exists a neighborhood $\mathcal{V} \subset C^{k+\ell+1}(S^1;\mathbb{R}^2)$ of the constant speed parameterized circle of radius $R > r/2$ such that the following hold:*

1. *For each $\gamma \in \mathcal{V}$ and $\varphi \in S^1$ the circle $B_r\big(\gamma(\varphi)\big)$ intersects the curve γ in exactly two points, denoted by $\gamma\big(p_\gamma(\varphi)\big)$ and $\gamma\big(m_\gamma(\varphi)\big)$. Here $m_\gamma(\varphi)$ denotes the previous intersection parameter and $p_\gamma(\varphi)$ the next one (see Figure 1).*
2. *The mappings*

$$\begin{aligned}m\colon C^{k+\ell+1}(S^1;\mathbb{R}^2) &\to \mathrm{Diff}^k(S^1)\,, & p\colon C^{k+\ell+1}(S^1;\mathbb{R}^2) &\to \mathrm{Diff}^k(S^1)\,,\\ \gamma &\mapsto m_\gamma\,, & \gamma &\mapsto p_\gamma\,,\end{aligned}$$

are ℓ-times continuously Fréchet differentiable. The first derivatives in direction $\sigma \in C^{k+\ell+1}(S^1;\mathbb{R}^2)$ are given by

$$p'_\gamma(\sigma) = \frac{\langle \sigma - \sigma(p_\gamma), \gamma(p_\gamma) - \gamma\rangle}{\langle \dot\gamma(p_\gamma), \gamma(p_\gamma) - \gamma\rangle}\,, \quad m'_\gamma(\sigma) = \frac{\langle \sigma - \sigma(m_\gamma), \gamma(m_\gamma) - \gamma\rangle}{\langle \dot\gamma(m_\gamma), \gamma(m_\gamma) - \gamma\rangle}\,.$$

Here $\mathrm{Diff}^k(S^1)$ denotes the group of C^k-diffeomorphisms on the unit circle.

Proof. Denote by $\gamma_0\colon S^1 \to \mathbb{R}^2$ the constant speed parameterized circle, that is, $\gamma_0(\varphi) = \big(R\cos(\varphi), R\sin(\varphi)\big)$. Then, for every $\varphi \in S^1$, the circle $B_r\big(\gamma_0(\varphi)\big)$ intersects γ_0 precisely at the two points $\big(R\cos(\varphi \pm \vartheta), R\sin(\varphi \pm \vartheta)\big)$, where

$$\vartheta = \arccos\left(1 - \frac{r^2}{2R^2}\right)\,.$$

Now define the mapping

$$\begin{aligned}F\colon C^{k+\ell+1}(S^1;\mathbb{R}^2) \times \mathrm{Diff}^k(S^1) &\to C^k(S^1;\mathbb{R})\,,\\ (\gamma, d) &\mapsto |\gamma(d) - \gamma|^2 - R^2\,.\end{aligned}$$

Obviously, the mappings p_γ and m_γ we are searching for satisfy $F(\gamma, p_\gamma) = 0$ and $F(\gamma, m_\gamma) = 0$. In particular, the equation $F(\gamma_0, d) = 0$ has the two solutions $p_{\gamma_0}(\varphi) := \varphi + \vartheta$ and $m_{\gamma_0}(\varphi) := \varphi - \vartheta$. Now, Lemma 3.3 implies that the mapping $\mathrm{Comp}\colon C^{k+\ell+1}(S^1;\mathbb{R}^2) \times \mathrm{Diff}^k(S^1) \to C^k(S^1;\mathbb{R}^2)$, and consequently also F, is ℓ-times continuously Fréchet differentiable. Moreover, it is easy to see that the derivative of F at (γ_0, p_{γ_0}) in direction $(0,\tau) \in C^{k+\ell+1}(S^1;\mathbb{R}^2) \times C^k(S^1;\mathbb{R})$ is given as

$$F'(\gamma_0, p_{\gamma_0})(0,\tau) = 2\langle \gamma_0(p_{\gamma_0}) - \gamma_0, \dot\gamma_0(p_{\gamma_0})\rangle \tau = 2\sin(\vartheta)\tau\,,$$

which is obviously an isomorphism of $C^k(S^1;\mathbb{R})$, as the assumption $R > r/2 > 0$ implies that $\sin(\vartheta) \neq 0$. Thus the implicit function theorem on Banach spaces (see [9, Sec. I.5]) implies the existence of a neighborhood $\mathcal{V} \subset C^{k+\ell+1}(S^1;\mathbb{R}^2)$ of γ_0 and unique ℓ-times continuously Fréchet differentiable mappings m, $p\colon \mathcal{V} \to \mathrm{Diff}^k(S^1)$ satisfying the equations $F(\gamma, p_\gamma) = 0 = F(\gamma, m_\gamma)$. The formula for the directional derivative of p at γ in direction σ now follows from the fact that

$$0 = \partial_\gamma F(\gamma, p_\gamma)(\sigma) = 2\langle \gamma(p_\gamma) - \gamma, \sigma(p_\gamma) - \sigma\rangle + 2p'_\gamma(\sigma)\langle \gamma(p_\gamma) - \gamma, \dot\gamma(p_\gamma)\rangle\,,$$

which is a simple application of the chain rule. The formula for $m'_\gamma(\sigma)$ can be derived analogously. □

Theorem 3.5. *For each $k \geq 0$ and $\ell \geq 1$ there exists a neighborhood $\mathcal{V} \subset C^{k+\ell+1}(S^1;\mathbb{R}^2)$ of the circle of radius $R > r/2$ such that the following hold:*

1. *For each $\gamma \in \mathcal{V}$ the circular integral invariant $I_r[\gamma]$ can be written as*

$$I_r[\gamma](\varphi) = \frac{1}{2}\int_{m_\gamma}^{p_\gamma} \langle \gamma(\psi) - \gamma, \dot\gamma(\psi)^\perp\rangle d\psi + \frac{r^2}{2}\arccos\left(\frac{\langle \gamma(p_\gamma) - \gamma, \gamma(m_\gamma) - \gamma\rangle}{r^2}\right). \quad (1)$$

2. *The circular integral invariant*

$$I_r\colon \mathcal{V} \to C^k(S^1;\mathbb{R})\,,$$
$$\gamma \mapsto I_r[\gamma]\,,$$

is ℓ-times continuously Fréchet differentiable. Its derivative in direction $\sigma \in C^{k+\ell+1}(S^1;\mathbb{R}^2)$ is given by

$$\begin{aligned}
&2I'_r[\gamma](\sigma)\\
&= 2\int_{m_\gamma}^{p_\gamma} \langle \sigma(\psi), \dot\gamma(\psi)^\perp\rangle d\psi\\
&\quad - \langle \sigma, \gamma(p_\gamma)^\perp - \gamma(m_\gamma)^\perp\rangle + \langle \gamma(p_\gamma) - \gamma, \sigma(p_\gamma)^\perp\rangle - \langle \gamma(m_\gamma) - \gamma, \sigma(m_\gamma)^\perp\rangle\\
&\quad + \langle \gamma(p_\gamma) - \gamma, \dot\gamma(p_\gamma)^\perp\rangle \frac{\langle \sigma - \sigma(p_\gamma), \gamma(p_\gamma) - \gamma\rangle}{\langle \dot\gamma(p_\gamma), \gamma(p_\gamma) - \gamma\rangle}\\
&\quad - \langle \gamma(m_\gamma) - \gamma, \dot\gamma(m_\gamma)^\perp\rangle \frac{\langle \sigma - \sigma(m_\gamma), \gamma(m_\gamma) - \gamma\rangle}{\langle \dot\gamma(m_\gamma), \gamma(m_\gamma) - \gamma\rangle}\\
&\quad - \frac{r^2}{\sqrt{r^4 - \langle \gamma(p_\gamma) - \gamma, \gamma(m_\gamma) - \gamma\rangle^2}}\cdot\\
&\qquad \cdot\Bigg(\frac{\langle \sigma - \sigma(p_\gamma), \gamma(p_\gamma) - \gamma\rangle}{\langle \dot\gamma(p_\gamma), \gamma(p_\gamma) - \gamma\rangle}\langle \dot\gamma(p_\gamma), \gamma(m_\gamma) - \gamma\rangle\\
&\qquad\quad + \langle \sigma(p_\gamma) - \sigma, \gamma(m_\gamma) - \gamma\rangle + \langle \gamma(p_\gamma) - \gamma, \sigma(m_\gamma) - \sigma\rangle\\
&\qquad\quad + \frac{\langle \sigma - \sigma(m_\gamma), \gamma(m_\gamma) - \gamma\rangle}{\langle \dot\gamma(m_\gamma), \gamma(m_\gamma) - \gamma\rangle}\langle \gamma(p_\gamma) - \gamma, \dot\gamma(m_\gamma)\rangle\Bigg)\,.
\end{aligned}$$

Proof. Under the given assumptions for R, r and $\mathcal{V}$, Formula (1) can be easily deduced from Figure 1. A term by term investigation of Formula (1), using Lemma 3.3 and Lemma 3.4, shows that I_r is of class C^ℓ on $\mathcal{V}$.

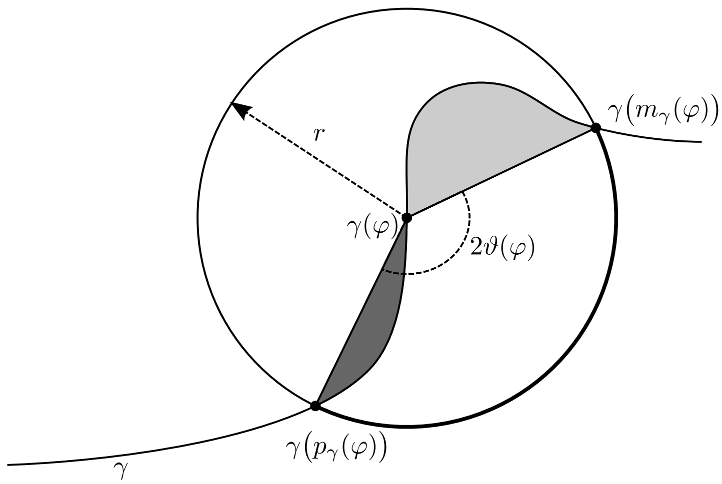

FIGURE 1. Sketch of the derivation of the analytical formula for the circular integral invariant assuming two points of intersection.

To calculate the differential of I_r we treat the two terms of Formula (1) separately. For the first term we obtain

$$\begin{aligned}
&\partial_\gamma \int_{m_\gamma}^{p_\gamma} \langle \gamma(\psi)-\gamma, \dot\gamma(\psi)^\perp\rangle d\psi \\
&\quad= \int_{m_\gamma}^{p_\gamma} \langle \sigma(\psi)-\sigma, \dot\gamma(\psi)^\perp\rangle + \langle \gamma(\psi)-\gamma, \dot\sigma(\psi)^\perp\rangle d\psi \\
&\qquad + \langle \gamma(p_\gamma)-\gamma, \dot\gamma(p_\gamma)^\perp\rangle p_\gamma'(\sigma) - \langle \gamma(m_\gamma)-\gamma, \dot\gamma(m_\gamma)^\perp\rangle m_\gamma'(\sigma) \\
&\quad= \int_{m_\gamma}^{p_\gamma} \langle \sigma(\psi)-\sigma, \dot\gamma(\psi)^\perp\rangle d\psi - \int_{m_\gamma}^{p_\gamma} \langle \dot\gamma(\psi), \sigma(\psi)^\perp\rangle d\psi \\
&\qquad + \langle \gamma(p_\gamma)-\gamma, \sigma(p_\gamma)^\perp\rangle - \langle \gamma(m_\gamma)-\gamma, \sigma(m_\gamma)^\perp\rangle \\
&\qquad + \langle \gamma(p_\gamma)-\gamma, \dot\gamma(p_\gamma)^\perp\rangle p_\gamma'(\sigma) - \langle \gamma(m_\gamma)-\gamma, \dot\gamma(m_\gamma)^\perp\rangle m_\gamma'(\sigma) \\
&\quad= \int_{m_\gamma}^{p_\gamma} 2\langle \sigma(\psi), \dot\gamma(\psi)^\perp\rangle d\psi - \langle \sigma, \gamma(p_\gamma)^\perp - \gamma(m_\gamma)^\perp\rangle \\
&\qquad + \langle \gamma(p_\gamma)-\gamma, \sigma(p_\gamma)^\perp\rangle - \langle \gamma(m_\gamma)-\gamma, \sigma(m_\gamma)^\perp\rangle \\
&\qquad + \langle \gamma(p_\gamma)-\gamma, \dot\gamma(p_\gamma)^\perp\rangle p_\gamma'(\sigma) - \langle \gamma(m_\gamma)-\gamma, \dot\gamma(m_\gamma)^\perp\rangle m_\gamma'(\sigma)\,.
\end{aligned}$$

A simple application of the chain rule yields for the second term

$$\begin{aligned}
&-\partial_\gamma \frac{r^2}{2} \arccos\left(\frac{\langle \gamma(p_\gamma)-\gamma, \gamma(m_\gamma)-\gamma\rangle}{r^2}\right) \\
&= \frac{\langle \dot\gamma(p_\gamma)p_\gamma'(\sigma)+\sigma(p_\gamma)-\sigma, \gamma(m_\gamma)-\gamma\rangle + \langle \gamma(p_\gamma)-\gamma, \dot\gamma(m_\gamma)m_\gamma'(\sigma)+\sigma(m_\gamma)-\sigma\rangle}{2r^{-2}\sqrt{r^4-\langle \gamma(p_\gamma)-\gamma, \gamma(m_\gamma)-\gamma\rangle^2}}\,.
\end{aligned}$$

Using the formulas for the intersection parameters, we obtain the desired result. $\square$

In the special case where γ equals the unit circle the lemma above reduces to:

Lemma 3.6. *Let $\gamma \in C^{k+\ell+1}(S^1;\mathbb{R}^2)$, $k \geq 0$, $\ell \geq 1$, be the constant speed parameterized unit circle, that is,*

$$\gamma(\varphi) = \big(\cos(\varphi), \sin(\varphi)\big)\ ,$$

and let $r < 2$. Then the derivative of $I_r[\gamma]$ in direction $\sigma \in C^{k+\ell+1}(S^1;\mathbb{R}^2)$ with

$$\sigma(\varphi) = a(\varphi)\dot\gamma(\varphi)^\perp + b(\varphi)\dot\gamma(\varphi)$$

is given by

$$I_r'[\gamma](\sigma)(\varphi) = \int_{\varphi-\vartheta}^{\varphi+\vartheta} a(\psi)\,d\psi - 2\sin(\vartheta)a(\varphi) = \big(\chi_{[-\vartheta,\vartheta]} * a\big)(\varphi) - 2\sin(\vartheta)a(\varphi)$$

with

$$\vartheta := \arccos\left(1 - \frac{r^2}{2}\right)\ .$$

The proof of this lemma is postponed to the appendix.

4. **Proof of the Main Theorem.**

Proof. We have already shown in Section 3 that I_r is of class C^ℓ.

We now show the local injectivity of I_r'. Without loss of generality we may assume that $R = 1$. Let $\mathcal{V}$ be the neighborhood of the unit circle defined in Theorem 3.5. The formula for I_r' implies that for every $\gamma \in \mathcal{V} \subset C^{k+\ell+1}(S^1;\mathbb{R}^2)$ the mapping $I_r'[\gamma]$ is bounded as a mapping from $C^k(S^1;\mathbb{R}^2)$ to $C^k(S^1;\mathbb{R})$. Thus, $I_r'[\gamma]$ has a unique bounded extension $J_r[\gamma]\colon C^k(S^1;\mathbb{R}^2) \to C^k(S^1;\mathbb{R})$. Moreover, the mapping J_r is continuous with respect to the $C^{k+\ell+1}$-topology seen as a mapping from $\mathcal{V}$ to $L\big(C^k(S^1;\mathbb{R}^2), C^k(S^1;\mathbb{R})\big)$. In addition, we denote for $\gamma \in \mathcal{V}$ by $\tilde J_r[\gamma]$ the restriction of $J_r[\gamma]$ to $T_\gamma\mathcal{C}_k$.

Denote now by $\gamma_0\colon S^1 \to \mathbb{R}^2$ the constant speed parameterized unit circle. Define for given $\sigma \in T_{\gamma_0}\mathcal{C}_k$ the function $A\sigma\colon S^1 \to \mathbb{R}$ by

$$A\sigma(\varphi) := \langle \sigma(\varphi), \dot\gamma_0(\varphi)^\perp\rangle\ .$$

Because γ_0 is a C^∞-curve, it follows that $A\sigma$ is C^k. Using Lemma 2.4 it follows that $A\sigma(0) = 0$ and $\partial_\varphi(A\sigma)(0) = 0$. Define now the space

$$\mathcal{A}^k(S^1;\mathbb{R}) := \big\{a \in C^k(S^1;\mathbb{R}) : a(0) = \dot a(0) = 0\big\}.$$

Then it follows that A is a bounded linear mapping from $T_{\gamma_0}\mathcal{C}_k$ to $\mathcal{A}^k(S^1;\mathbb{R})$.

In addition, it follows from Lemma 2.4 that A is boundedly invertible with A^{-1} given by

$$A^{-1}a = a\dot\gamma_0^\perp + b\dot\gamma_0$$

with

$$b(\varphi) = \dot b(0)\varphi + \int_0^\varphi a(\tau)\,d\tau \quad\text{ and }\quad \dot b(0) = \frac{1}{2\pi}\int_0^{2\pi} a(\tau)\,d\tau\ .$$

The expression for $\dot b(0)$ is due to the periodicity of b, which implies that

$$0 = b(0) = b(2\pi) = 2\pi\dot b(0) + \int_0^{2\pi} a(\tau)\,d\tau\ .$$

Therefore A is in fact an isomorphism between $T_{\gamma_0}\mathcal{C}_k$ and $\mathcal{A}^k(S^1;\mathbb{R})$.

According to Lemma 3.6, the mapping $\tilde{J}_r[\gamma_0]$ evaluated at $\sigma = a\dot{\gamma}_0^{\perp} + b\dot{\gamma}_0 \in T_{\gamma_0}\mathcal{C}_k$ can be written as

$$\tilde{J}_r[\gamma_0](\sigma) = \chi_{[-\vartheta,\vartheta]} * a - 2\sin(\vartheta)a \,.$$

Thus $\tilde{J}_r[\gamma_0]$ can be decomposed into

$$\tilde{J}_r[\gamma_0] = B \circ \imath \circ A \,,$$

where the operator $B\colon C^k(S^1;\mathbb{R}) \to C^k(S^1;\mathbb{R})$ is given by

$$Ba = \chi_{[-\vartheta,\vartheta]} * a - 2\sin(\vartheta)a$$

and $\imath$ is the embedding from $\mathcal{A}^k(S^1;\mathbb{R})$ into $C^k(S^1;\mathbb{R})$. Lemma 6.1 (see Appendix) implies that the mapping $\sigma \mapsto \chi_{[-\vartheta,\vartheta]} * a$ is compact and thus B is a compact perturbation of the identity. Therefore the Riesz–Schauder theory (see [17, Chap. X.5]) implies that B has a closed range.

Next we compute the kernel of B. To that end we consider the mapping in the Fourier basis. A short calculation shows that in this basis the operator B is the diagonal operator that maps a sequence of (complex) Fourier coefficients $(c_j)_{j\in\mathbb{Z}}$ to the sequence $(d_jc_j)_{j\in\mathbb{Z}}$, where

$$d_j = \begin{cases} 2\bigl(1-\sin(\vartheta)\bigr) & \text{if } j = 0\,, \\ 0 & \text{if } j = \pm 1\,, \\ \Bigl[2\frac{\sin(j\vartheta)}{j} - 2\sin(\vartheta)\Bigr] & \text{else.} \end{cases}$$

Because $\sin(\vartheta) \neq 1$ and $\sin(j\vartheta) \neq j\sin(\vartheta)$ whenever $j \in \mathbb{Z}\setminus\{-1,0,1\}$ (see Lemma 6.2 in the Appendix), it follows that the kernel of B consists of the functions a of the form $a(\varphi) = c_{-1}\exp(-i\varphi) + c_1\exp(i\varphi)$ for some c_{-1}, $c_1 \in \mathbb{C}$.

In the next step we show that the kernel of $\tilde{J}_r[\gamma_0] = B\circ\imath\circ A$ is trivial. Therefore assume that $a = c_{-1}\exp(-i\cdot) + c_1\exp(i\cdot) \in \mathcal{A}^k(S^1;\mathbb{R}) \cap \operatorname{Ker} B$. Because $a(0) = 0$, it follows that $c_{-1} + c_1 = 0$; because $\dot{a}(0) = 0$, it follows that $-c_{-1} + c_1 = 0$. Together, this shows that $c_{-1} = c_1 = 0$, implying that the intersection of $\operatorname{Ker} B$ with $\mathcal{A}^k(S^1;\mathbb{R})$ is trivial. Since A is an isomorphism this proves the injectivity of $\tilde{J}_r[\gamma_0]$.

We have thus shown that $\tilde{J}_r[\gamma_0] = B\circ\imath\circ A$ is strongly closed and injective. Now note that the set of strongly closed and injective, bounded linear functionals between two Banach spaces X and Y is open with respect to the norm topology on $L(X,Y)$. Because of the continuity of J_r and therefore $\tilde{J}_r$, this proves the existence of a neighborhood $\gamma_0 \in \mathcal{U} \subset \mathcal{C}_{k+2}$ such that $\tilde{J}_r[\gamma]$ is injective for every $\gamma \in \mathcal{U}$. In particular, this proves the injectivity of $I_r'[\gamma]$ for every $\gamma \in \mathcal{U}$.

For proving the local injectivity, let γ, $\tilde{\gamma} \in \mathcal{C}_{k+6} \cap \mathcal{U}$. Because the mapping $I_r\colon \mathcal{V} \subset C^{k+3}(S^1;\mathbb{R}^2) \to C^k(S^1;\mathbb{R})$ is of class C^2, it has a Taylor expansion of the form

$$I_r[\tilde{\gamma}] = I_r[\gamma] + I_r'[\gamma](\tilde{\gamma}-\gamma) + \int_0^1 (1-t)I_r''[\gamma + t(\tilde{\gamma}-\gamma)](\tilde{\gamma}-\gamma,\tilde{\gamma}-\gamma)\,dt.$$

Thus

$$\begin{aligned} \|I_r[\tilde{\gamma}] - I_r[\gamma]\|_k &\geq \|I_r'[\gamma](\tilde{\gamma}-\gamma)\|_k - \Bigl\|\int_0^1 (1-t)I_r''[\gamma + t(\tilde{\gamma}-\gamma)](\tilde{\gamma}-\gamma,\tilde{\gamma}-\gamma)\,dt\Bigr\|_k \\ &\geq \|I_r'[\gamma](\tilde{\gamma}-\gamma)\|_k - \int_0^1 \Bigl\|I_r''[\gamma + t(\tilde{\gamma}-\gamma)](\tilde{\gamma}-\gamma,\tilde{\gamma}-\gamma)\Bigr\|_k\,dt \end{aligned}$$

Because $I_r\colon C^{k+3}\to C^k$ is of class C^2, it follows that $I_r''\colon C^{k+3}\to L^2(C^{k+3},C^k)$ is continuous. Thus there exists a convex neighborhood $\mathcal{V}_1$ of the circle with respect to the C^{k+3}-norm and a constant c_1 such that $\|I_r''[\widehat{\gamma}]\|_{L^2(C^{k+3},C^k)}\le c_1$ for every $\widehat{\gamma}\in\mathcal{V}_1$. Consequently,

$$\|I_r[\tilde\gamma]-I_r[\gamma]\|_k \ge \|I_r'[\gamma](\tilde\gamma-\gamma)\|_k - c_1\|\tilde\gamma-\gamma\|_{k+3}^2 \tag{2}$$

for every γ, $\tilde\gamma\in\mathcal{V}_1$.

Since I_r' can be extended to a continuous mapping $J_r\colon\mathcal{V}\to L(C^k,C^k)$, for every $\varepsilon>0$ there exists a neighbourhood $\mathcal{V}_\varepsilon$ of the constant speed parameterized unit circle γ_0 such that $\|J_r[\gamma]-J_r[\gamma_0]\|<\varepsilon$ for every $\gamma\in\mathcal{V}_\varepsilon$. In particular, we have for $\gamma\in\mathcal{V}_\varepsilon$

$$\begin{aligned}\|I_r'[\gamma](\tilde\gamma-\gamma)\|_k &= \|J_r[\gamma](\tilde\gamma-\gamma)\|_k\\ &\ge \|J_r[\gamma_0](\tilde\gamma-\gamma)\|_k-\varepsilon\|\tilde\gamma-\gamma\|_k\\ &= \|I_r'[\gamma_0](\tilde\gamma-\gamma)\|_k-\varepsilon\|\tilde\gamma-\gamma\|_k.\end{aligned}\tag{3}$$

Since $\mathcal{C}_k$ is a smooth submanifold of $C^k(S^1;\mathbb{R}^2)$, there exists a neighborhood $\mathcal{V}_2\subset C^k(S^1;\mathbb{R}^2)$ of γ_0 and a smooth diffeomorphism $\Phi\colon\mathcal{V}_2\to\Phi(\mathcal{V}_2)\subset C^k(S^1;\mathbb{R}^2)$ such that

$$\Phi[\gamma_0]=0,\qquad \Phi(\mathcal{V}_2\cap\mathcal{C}_k)\subset T_{\gamma_0}\mathcal{C}_k,\qquad\text{and}\qquad \Phi'[\gamma_0]=\operatorname{Id}.$$

In order to show that such a map exists, let $\Psi\colon\mathcal{V}_2\subset C^k(S^1;\mathbb{R}^2)\to C^k(S^1;\mathbb{R}^2)$ be any submanifold chart centered at γ_0. That is, there exists a closed linear subspace $\mathcal{E}\subset C^k(S^1;\mathbb{R}^2)$ such that $\Psi(\mathcal{V}_2\cap\mathcal{C}_k)=\Psi(\mathcal{V}_2)\cap\mathcal{E}$. Then the mapping $\Phi:=\Psi'[\gamma_0]^{-1}\circ\Psi$ has the desired properties.

Thus the continuous invertibility of $I_r'[\gamma_0]$ on $T_{\gamma_0}\mathcal{C}_k$ seen as a mapping from C^k to C^k implies that there exists $c_2>0$ such that for every γ, $\tilde\gamma\in\mathcal{V}_2\cap\mathcal{C}_k$ we have

$$\begin{aligned}\|I_r'[\gamma_0](\tilde\gamma-\gamma)\|_k &\ge \|I_r'[\gamma_0](\Phi(\tilde\gamma)-\Phi(\gamma))\|_k-\|I_r'[\gamma_0](\tilde\gamma-\gamma-\Phi(\tilde\gamma)+\Phi(\gamma))\|_k\\ &\ge c_2\|\Phi(\tilde\gamma)-\Phi(\gamma)\|_k-c_3\|\tilde\gamma-\gamma-\Phi(\tilde\gamma)+\Phi(\gamma)\|_k\end{aligned}\tag{4}$$

with $c_3:=\|I_r'[\gamma_0]\|_{k+3,k}$.

Developing $\Phi(\tilde\gamma)$ in a Taylor expansion at γ, we obtain after possibly choosing a smaller neighbourhood

$$\Phi(\tilde\gamma)=\Phi(\gamma)+\Phi'[\gamma](\tilde\gamma-\gamma)+R(\gamma,\tilde\gamma)$$

with

$$\|R(\gamma,\tilde\gamma)\|_k\le c_4\|\gamma-\tilde\gamma\|_k^2$$

for some $c_4>0$ independent of γ. Inserting this Taylor expansion into (4) yields

$$\begin{aligned}\|I_r'[\gamma_0](\tilde\gamma-\gamma)\|_k &\ge c_2\|\Phi'[\gamma](\tilde\gamma-\gamma)\|_k-c_3\|(\operatorname{Id}-\Phi'[\gamma])(\tilde\gamma-\gamma)\|_k\\ &\qquad-(c_2+c_3)\|R(\gamma,\tilde\gamma)\|_k\\ &\ge c_2\|\tilde\gamma-\gamma\|_k-(c_2+c_3)\|(\operatorname{Id}-\Phi'[\gamma])(\tilde\gamma-\gamma)\|_k\\ &\qquad-c_4(c_2+c_3)\|\gamma-\tilde\gamma\|_k^2\\ &\ge \big(c_2-c_5\|\operatorname{Id}-\Phi'[\gamma]\|_{k,k}-c_6\|\gamma-\tilde\gamma\|_k\big)\|\gamma-\tilde\gamma\|_k.\end{aligned}$$

Because Φ is smooth and $\Phi'[\gamma_0]=\operatorname{Id}$, it follows that there exists a neighbourhood $\mathcal{V}_3$ of γ_0 such that

$$\|I_r'[\gamma_0](\tilde\gamma-\gamma)\|_k\ge c_7\|\gamma-\tilde\gamma\|_k \tag{5}$$

for every γ, $\tilde\gamma\in\mathcal{V}_3\cap\mathcal{C}_k$.

Collecting the inequalities (2), (3), and (5), we obtain

$$\|I_r[\tilde\gamma] - I_r[\gamma]\|_k \geq (c_7 - \varepsilon)\|\gamma - \tilde\gamma\|_k - c_1\|\gamma - \tilde\gamma\|_{k+3}^2 \tag{6}$$

for every γ, $\tilde\gamma \in \mathcal{V}_1 \cap \mathcal{V}_\varepsilon \cap \mathcal{V}_2 \cap \mathcal{C}_{k+3}$.

In order to obtain the desired result, we use the interpolation inequality (see [6, Theorem 2.2.1, p. 143])

$$\|\gamma - \tilde\gamma\|_{k+3}^2 \leq c_8\|\gamma - \tilde\gamma\|_k\|\gamma - \tilde\gamma\|_{k+6}.$$

Choosing $\varepsilon > 0$ in (6) sufficiently small, we obtain the estimate

$$\|I_r[\tilde\gamma] - I_r[\gamma]\|_k \geq (c_9 - c_8\|\gamma - \tilde\gamma\|_{k+6})\|\gamma - \tilde\gamma\|_k$$

for every γ, $\tilde\gamma \in \mathcal{V}_1 \cap \mathcal{V}_\varepsilon \cap \mathcal{V}_2 \cap \mathcal{C}_{k+6}$. Thus there exists a neighbourhood $\tilde{\mathcal{U}} \subset \mathcal{C}_{k+6}$ of the circle γ_0 and a constant $C > 0$ such that

$$\|I_r[\tilde\gamma] - I_r[\gamma]\|_k \geq C\|\gamma - \tilde\gamma\|_k$$

for every γ, $\tilde\gamma \in \tilde{\mathcal{U}}$. □

5. Conclusion.

In this article, we have shown an injectivity result for the circular integral invariant on a C^{k+6} neighborhood $\tilde{\mathcal{U}}$ of the circle. Note, however, that the derived result does not prove the continuous invertibility of the invariant on $\tilde{\mathcal{U}}$.

The classical approach for proving such a result would be the usage of the inverse function theorem. To that end, however, we would require that the mapping I_r was continuously Fréchet differentiable and its derivative I_r' an isomorphism of the corresponding tangent spaces. Although our results prove that I_r' can be extended to an isomorphism $\tilde J_r$ of $T_{\gamma_0}\mathcal{C}_k$, we cannot use the inverse function theorem, as the mapping I_r is not Fréchet differentiable (and not even Gâteaux differentiable) from C^k to C^k — in fact, our results do not even prove that I_r maps C^k curves into C^k integral invariants. Conversely, seen as a mapping from C^{k+2} to C^k, the tangent mapping, though injective, cannot be a surjection near the circle.

Another issue are the rather stringent smoothness assumptions. We believe that it is possible to relax these assumptions, as it seems probable that I_r is twice weakly differentiable as a mapping from C^{k+2} to C^k, which would indicate a possible uniqueness result on a C^{k+4}-neighborhood of the circle.

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6. Appendix.

Lemma 6.1. *For every $k \geq 0$, the mapping $a \mapsto K_\vartheta a := \chi_{[-\vartheta,\vartheta]} * a$ is compact as a mapping from $C^k(S^1;\mathbb{R})$ to $C^k(S^1;\mathbb{R})$.*

Proof. Denoting by B_1 the unit ball in $C^k(S^1;\mathbb{R})$, we have to show that the image of B_1 under K_ϑ is precompact in $C^k(S^1;\mathbb{R})$. Applying the Arzelà–Ascoli Theorem (see [17, Chap. III.3]), we have to show that $K_\vartheta(B_1)$ is bounded and the first k derivatives of the functions in $K_\vartheta(B_1)$ are equicontinuous. The boundedness of $K_\vartheta(B_1)$ is obvious, K_ϑ being a bounded linear mapping (of norm 2ϑ). Now assume that $a \in B_1$. Then $\partial_\varphi^j(K_\vartheta a)(\varphi) = a^{(j-1)}(\varphi+\vartheta) - a^{(j-1)}(\varphi-\vartheta)$. Because $\|a^{(j)}\|_\infty \leq 1$ for all $1 \leq j \leq k$, it follows that $\partial_\varphi(K_\vartheta a)$ is Lipschitz continuous with Lipschitz constant at most 2. Hence $K_\vartheta(B_1)$ is a precompact set. □

Lemma 6.2. *Let $0 < \vartheta < \pi$ and $j \in \mathbb{Z} \setminus \{-1, 0, 1\}$. Then $\sin(j\vartheta) \neq j \sin(\vartheta)$.*

Proof. Assume first that $0 < \vartheta \leq \pi/2$. We show that in this case the equation $\sin(s) = s \sin(\vartheta)/\vartheta$ has the only solutions $s = 0$ and $s = \pm\vartheta$. First note that the strict concavity of the sine function on the interval $[0, \pi]$ implies that on this interval we only have two solutions, namely 0 and ϑ. Moreover, the concavity of the sine implies that $\sin(\vartheta)/\vartheta \geq \sin(\pi/2)/(\pi/2) = 2/\pi$, and therefore $\pi \sin(\vartheta)/\vartheta \geq 2$. This, however, implies that the equation $\sin(s) = s\sin(\vartheta)/\vartheta$ cannot have any solutions for $s > \pi$, as the right hand side is strictly larger than 2. The fact that $-\vartheta$ is the only negative solution follows by symmetry. In particular, setting $s = j\vartheta$, this proves the assertion in the case $0 < \vartheta \leq \pi/2$.

Now assume that $\pi/2 < \vartheta < \pi$ and let $\psi := \pi - \vartheta$. Then

$$\sin(\vartheta) = \sin(\pi - \psi) = \sin(\psi) .$$

Now, if j is odd, then

$$\sin(j\vartheta) = \sin(j\pi - j\psi) = \sin(\pi - j\psi) = \sin(j\psi) .$$

Thus $j \sin(\vartheta) = \sin(j\vartheta)$, if and only if $j \sin(\psi) = \sin(j\psi)$. Because $0 < \psi < \pi/2$, the first part of the proof can be applied, showing that $j \sin(\vartheta) \neq \sin(j\vartheta)$ unless $j = \pm 1$.

On the other hand, if j is even, we have

$$\sin(j\vartheta) = \sin(j\pi - j\psi) = \sin(-j\psi) = -\sin(j\psi) .$$

Thus, $j\sin(\vartheta) = \sin(j\vartheta)$, if and only if $j \sin(\psi) = -\sin(j\psi)$. Now note that the equation $-\sin(s) = s \sin(\psi)/\psi$ has only the trivial solution $s = 0$, because $\sin(\psi)/\psi \geq 2/\pi$ and the left hand side is negative for $0 < s < \pi$. As a consequence, the equation $j \sin(\psi) = -\sin(j\psi)$ only holds for $j = 0$, which concludes the proof. □

6.1. **Proof of Lemma 3.6.**

Proof. Let $\gamma \in C^{k+2}(S^1; \mathbb{R}^2)$ be the constant speed parameterized unit circle. Then

$$\gamma(\varphi) = \bigl(\cos(\varphi), \sin(\varphi)\bigr) , \qquad \dot\gamma(\varphi) = \bigl(-\sin(\varphi), \cos(\varphi)\bigr) ,$$
$$\gamma(\varphi)^\perp = \bigl(\sin(\varphi), -\cos(\varphi)\bigr) , \qquad \dot\gamma(\varphi)^\perp = \bigl(\cos(\varphi), \sin(\varphi)\bigr) .$$

Because γ is the unit circle, there exists $\vartheta \in S^1$ such that

$$p_\gamma(\varphi) = \varphi + \vartheta , \qquad m_\gamma(\varphi) = \varphi - \vartheta .$$

Obviously, all assumptions of Lemma 3.5 are satisfied. It remains to calculate all the terms that appear in the expression of the Fréchet derivative in Lemma 3.5 for the special case of the unit circle. In particular, we obtain

$$
\begin{aligned}
r^2 = \|\gamma(p_\gamma) - \gamma\|^2 &= 2\big(1 - \cos(\vartheta)\big)\,,\\
\langle \gamma(p_\gamma) - \gamma, \gamma(m_\gamma) - \gamma\rangle &= -2\cos(\vartheta)\big(1 - \cos(\vartheta)\big)\,,\\
\langle \dot\gamma(p_\gamma), \gamma(m_\gamma) - \gamma\rangle &= \sin(\vartheta)\big(1 - 2\cos(\vartheta)\big)\,,\\
\langle \dot\gamma(p_\gamma)^\perp, \gamma(p_\gamma) - \gamma\rangle &= 1 - \cos(\vartheta)\,,\\
\langle \dot\gamma(p_\gamma), \gamma(p_\gamma) - \gamma\rangle &= \sin(\vartheta)\,,\\
\langle \dot\gamma, \gamma(p_\gamma) - \gamma\rangle &= \sin(\vartheta)\,,\\
\langle \dot\gamma^\perp, \gamma(p_\gamma) - \gamma\rangle &= \cos(\vartheta) - 1\,,\\
\langle \dot\gamma(p_\gamma)^\perp, \gamma(m_\gamma) - \gamma\rangle &= \cos^2(\vartheta) - \cos(\vartheta) - \sin^2(\vartheta)\,,\\
\langle \dot\gamma^\perp, \gamma(m_\gamma) - \gamma\rangle &= \cos(\vartheta) - 1\,,\\
\langle \dot\gamma(m_\gamma), \gamma(p_\gamma) - \gamma\rangle &= -\sin(\vartheta)\big(1 - 2\cos(\vartheta)\big)\,,\\
\langle \dot\gamma(m_\gamma)^\perp, \gamma(m_\gamma) - \gamma\rangle &= 1 - \cos(\vartheta)\,,\\
\langle \dot\gamma(m_\gamma), \gamma(m_\gamma) - \gamma\rangle &= -\sin(\vartheta)\,,\\
\langle \dot\gamma, \gamma(m_\gamma) - \gamma\rangle &= -\sin(\vartheta)\,,\\
\langle \dot\gamma(m_\gamma)^\perp, \gamma(p_\gamma) - \gamma\rangle &= \cos^2(\vartheta) - \cos(\vartheta) - \sin^2(\vartheta)\,,\\
\langle \dot\gamma, \gamma(p_\gamma) - \gamma(m_\gamma)\rangle &= 2\sin(\vartheta)\;.
\end{aligned}
$$

In the following we treat the derivative in direction $a\dot\gamma^\perp$ and $b\dot\gamma$ separately. For the derivative in normal direction $a\dot\gamma^\perp$ we get

$$
\begin{aligned}
&2I_r'[\gamma](a\dot\gamma^\perp) = 2\int_{m_\gamma}^{p_\gamma} a(\psi)d\psi - a\langle \dot\gamma^\perp, \gamma(p_\gamma)^\perp - \gamma(m_\gamma)^\perp\rangle\\
&\quad - a(p_\gamma)\langle \gamma(p_\gamma) - \gamma, \dot\gamma(p_\gamma)\rangle + a(m_\gamma)\langle \gamma(m_\gamma) - \gamma, \dot\gamma(m_\gamma)\rangle\\
&\quad + \langle \gamma(p_\gamma) - \gamma, \dot\gamma(p_\gamma)^\perp\rangle \frac{\langle a\dot\gamma^\perp - a(p_\gamma)\dot\gamma(p_\gamma)^\perp, \gamma(p_\gamma) - \gamma\rangle}{\langle \dot\gamma(p_\gamma), \gamma(p_\gamma) - \gamma\rangle}\\
&\quad - \langle \gamma(m_\gamma) - \gamma, \dot\gamma(m_\gamma)^\perp\rangle \frac{\langle a\dot\gamma^\perp - a(m_\gamma)\dot\gamma(m_\gamma)^\perp, \gamma(m_\gamma) - \gamma\rangle}{\langle \dot\gamma(m_\gamma), \gamma(m_\gamma) - \gamma\rangle}\\
&\quad - \frac{r^2}{\sqrt{r^4 - \langle \gamma(p_\gamma) - \gamma, \gamma(m_\gamma) - \gamma\rangle^2}}\cdot\\
&\qquad \cdot\Bigg(\frac{\langle a\dot\gamma^\perp - a(p_\gamma)\dot\gamma(p_\gamma)^\perp, \gamma(p_\gamma) - \gamma\rangle}{\langle \dot\gamma(p_\gamma), \gamma(p_\gamma) - \gamma\rangle}\langle \dot\gamma(p_\gamma), \gamma(m_\gamma) - \gamma\rangle\\
&\qquad\quad + \langle a(p_\gamma)\dot\gamma(p_\gamma)^\perp - a\dot\gamma^\perp, \gamma(m_\gamma) - \gamma\rangle + \langle \gamma(p_\gamma) - \gamma, a(m_\gamma)\dot\gamma(m_\gamma)^\perp - a\dot\gamma^\perp\rangle\\
&\qquad\quad + \frac{\langle a\dot\gamma^\perp - a(m_\gamma)\dot\gamma(m_\gamma)^\perp, \gamma(m_\gamma) - \gamma\rangle}{\langle \dot\gamma(m_\gamma), \gamma(m_\gamma) - \gamma\rangle}\langle \gamma(p_\gamma) - \gamma, \dot\gamma(m_\gamma)\rangle\Bigg)\;.
\end{aligned}
$$

Inserting the expressions for γ equal to the unit circle that have been calculated previously, we obtain

$$
\begin{aligned}
&2I_r'[\gamma](a\dot\gamma^\perp)\\
&= 2\int_{\varphi-\vartheta}^{\varphi+\vartheta} a(\psi)d\psi\\
&\qquad - \sin(\vartheta)\big(a(p_\gamma)+2a+a(m_\gamma)\big) - \frac{(\cos(\vartheta)-1)^2}{\sin(\vartheta)}\big(a(p_\gamma)+2a+a(m_\gamma)\big)\\
&\qquad - \frac{1}{\sin(\vartheta)}\Big(\big(a+a(p_\gamma)\big)\big(\cos(\vartheta)-1\big)\big(1-2\cos(\vartheta)\big) - 2a(\cos(\vartheta)-1)\\
&\qquad\quad + \big(a(p_\gamma)+a(m_\gamma)\big)\big(\cos^2(\vartheta)-\cos(\vartheta)-\sin^2(\vartheta)\big)\\
&\qquad\quad + \big(a+a(m_\gamma)\big)\big(\cos(\vartheta)-1\big)\big(1-2\cos(\vartheta)\big)\Big)\\
&= 2\int_{\varphi-\vartheta}^{\varphi+\vartheta} a(\psi)d\psi\\
&\qquad + \frac{\cos(\vartheta)-1}{\sin(\vartheta)}\big(2a(p_\gamma)+4a+2a(m_\gamma)-2a(p_\gamma)+4\cos(\vartheta)a-2a(m_\gamma)\big)\\
&= 2\int_{\varphi-\vartheta}^{\varphi+\vartheta} a(\psi)d\psi + 4a\frac{(\cos(\vartheta)-1)(\cos(\vartheta)+1)}{\sin(\vartheta)}\\
&= 2\int_{\varphi-\vartheta}^{\varphi+\vartheta} a(\psi)d\psi - 4a\sin(\vartheta)\ .
\end{aligned}
$$

Inserting the formulas above in the expression for $I_r'[\gamma](b\dot\gamma)$ yields

$$
\begin{aligned}
&2I_r'[\gamma](b\dot\gamma) = 2\int_{m_\gamma}^{p_\gamma} b(\psi)\langle\dot\gamma(\psi),\dot\gamma(\psi)^\perp\rangle d\psi\\
&\quad - \langle b\dot\gamma, \gamma(p_\gamma)^\perp - \gamma(m_\gamma)^\perp\rangle + \langle\gamma(p_\gamma)-\gamma, b(p_\gamma)\dot\gamma(p_\gamma)^\perp\rangle\\
&\quad - \langle\gamma(m_\gamma)-\gamma, b(m_\gamma)\dot\gamma(m_\gamma)^\perp\rangle + \langle\gamma(p_\gamma)-\gamma,\dot\gamma(p_\gamma)^\perp\rangle\frac{\langle b\dot\gamma - b(p_\gamma)\dot\gamma(p_\gamma),\gamma(p_\gamma)-\gamma\rangle}{\langle\dot\gamma(p_\gamma),\gamma(p_\gamma)-\gamma\rangle}\\
&\quad - \langle\gamma(m_\gamma)-\gamma,\dot\gamma(m_\gamma)^\perp\rangle\frac{\langle b\dot\gamma - b(m_\gamma)\dot\gamma(m_\gamma),\gamma(m_\gamma)-\gamma\rangle}{\langle\dot\gamma(m_\gamma),\gamma(m_\gamma)-\gamma\rangle}\\
&\quad - \frac{r^2}{\sqrt{r^4-\langle\gamma(p_\gamma)-\gamma,\gamma(m_\gamma)-\gamma\rangle^2}}\cdot\\
&\qquad\cdot\Bigg(\frac{\langle b\dot\gamma - b(p_\gamma)\dot\gamma(p_\gamma),\gamma(p_\gamma)-\gamma\rangle}{\langle\dot\gamma(p_\gamma),\gamma(p_\gamma)-\gamma\rangle}\langle\dot\gamma(p_\gamma),\gamma(m_\gamma)-\gamma\rangle\\
&\qquad\quad + \langle b(p_\gamma)\dot\gamma(p_\gamma)-b\dot\gamma,\gamma(m_\gamma)-\gamma\rangle + \langle\gamma(p_\gamma)-\gamma, b(m_\gamma)\dot\gamma(m_\gamma)-b\dot\gamma\rangle\\
&\qquad\quad + \frac{\langle b\dot\gamma - b(m_\gamma)\dot\gamma(m_\gamma),\gamma(m_\gamma)-\gamma\rangle}{\langle\dot\gamma(m_\gamma),\gamma(m_\gamma)-\gamma\rangle}\langle\gamma(p_\gamma)-\gamma,\dot\gamma(m_\gamma)\rangle\Bigg)\\
&= 0 - 0 + \big(1-\cos(\vartheta)\big)\big(b(p_\gamma)-b(m_\gamma)+b-b(p_\gamma)-b+b(m_\gamma)\big)\\
&\quad - \Big(\big(1-2\cos(\vartheta)\big)\big(b-b(p_\gamma)+b(p_\gamma)-b(m_\gamma)+b(m_\gamma)-b\big)+(b-b)\Big)\\
&= 0\ .
\end{aligned}
$$

Therefore,

$$I_r'[\gamma](a\dot\gamma^\perp + b\dot\gamma) = \chi_{[-\vartheta,\vartheta]} * a - 2\sin(\vartheta)a\ . \qquad \square$$

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