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Applications of Reproducing Kernel Hilbert Spaces—Bandlimited Signal Models*

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The finite energy Fourier-, Hankel-, sine-, and cosine-transformed bandlimited signals are specific realizations of the abstract reproducing kernel Hilbert space (RKHS). Basic properties of the abstract RKHS are applied to the detailed study of bandlimited signals. The relevancy of the reproducing kernel in extremum problems is discussed. New and known results in sampling expansions, minimum energy and non-uniform interpolations, and truncation error bounds are presented from a unified point of view of the RKHS. Some generalizations and extensions are stated.

LIST OF SYMBOLS

H, H_i	Reproducing Kernel Hilbert Spaces (RKHS)
H', H''	Subspaces of RKHS
x, f, f_0, g, h	Functions in RKHS
K, K_i	Reproducing kernels
Y, Ω	Spaces of functions
T	Index set (continuum; time)
s, t, t_i	Elements in T
$F(\omega)$	Function in the transformed domain
$M, M_i, E, C, A, c_i, c, m_i$	Constants
G_i, D_i, d_i	$n \times n$ determinants
$\mathbf{K}$	$n \times n$ matrix
λ, λ_j	Eigenvalues
$\boldsymbol{\theta}, \boldsymbol{\theta}_j, \hat{\mathbf{f}}$	$n \times 1$ column vectors
$\boldsymbol{\theta}^T, \boldsymbol{\theta}_j^T, \mathbf{f}^T$	$1 \times n$ row vector
$\theta_{j,i}$	i^{th} element of $\boldsymbol{\theta}_j$
ϕ_i	i^{th} element of a c.o.n. system

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ψ_i	i^{th} sampling function
J_ν	Bessel function of the first kind and order ν, $\nu \geqq -\frac{1}{2}$
L	Bound linear functional
I, I'	Index set (integral)
$i_0(s)$	Nearest integer to s
$E_{I'}$	Truncation error

In communication and information theories, bandlimited (deterministic and random) signal models are used for analysis and representations. These models are used because often they represent fairly well the actual signals encountered in practice. Furthermore, many mathematical properties can and have been derived from these models. This paper deals with some applications of reproducing kernel Hilbert space methods to bandlimited signal models.

The basic mathematical properties of the reproducing kernel Hilbert space (henceforth abbreviated as RKHS) were studied by Moore (1935), Bergman (1950), and Aronszajn (1950). Applications of RKHS methods to second-order stochastic processes were given by Loève (1948). RKHS methods have been found useful in time series, detection, filtering, and prediction problems. (Parzen, 1961, 1962 and Kailath, 1967).

In Section I, two equivalent definitions of the abstract RKHS are stated. Finite-energy, bandlimited signals associated with Fourier-, Hankel-, sine-, and cosine-transforms are shown to be specific realizations of the abstract RKHS.

In Section II, the relevancy of the reproducing kernel in some extremum problems is discussed. For a given single sampling instant, t, the minimum energy signal that satisfies an interpolation requirement at t and the signals with a given upper energy bound that maximize the square of the signals' value at t are easily obtained. When there are n distinct sampling instants, the solutions to these two problems are obtained from orthonormalization and the solution of a classical eigenvalue equation involving the reproducing kernel indexed by the sampling instants.

In Section III, various properties of sampling expansions in RKHS are discussed. A simple relationship between sampling expansion and complete orthonormal expansion is obtained. Sampling expansions for the four classes of bandlimited signals are stated. General as well as specific (Shannon sampling expansion) truncation error bounds are calculated. Finally, necessary and sufficient conditions are given for a finite expansion of the reproducing kernel indexed by the sampling instants to satisfy the minimum energy and interpolation propetries.

The results in Sections I, II, and III are often stated in the simplest. and not in the most general, terms. Various generalizations and extensions are discussed in Section IV.

I. PRELIMINARY

A Hilbert space is a complete infinite-dimensional inner-product space. The elements of this space can be functions defined on a set T. In particular, the abstract reproducing kernel Hilbert space (RKHS), H, is a Hilbert space of functions defined on a set T such that there exists a unique function, $K(s, t)$, defined on $T \times T$ with the following properties:

$$K(\cdot, t) \in H, \qquad \forall t \in T. \tag{1}$$

$$x(t) = (x, K(\cdot, t)), \qquad \forall t \in T, \quad \forall x \in H. \tag{2}$$

The function $K(s, t)$ is called the reproducing kernel of the abstract RKHS. (Aronszajn, 1950).

An equivalent definition of the abstract RKHS can also be given. The abstract proper functional Hilbert space, H, is a Hilbert space of functions defined on a set T such that the linear functional, $x(t)$, is bounded for every $x \in H$ and every $t \in T$. From the Riesz linear functional representation theorem, it is clear that the abstract reproducing kernel Hilbert space is equivalent to the abstract proper functional Hilbert space. For simplicity, we assume the scalars and the functions are all real-valued in the RKHS.

In signal analysis, the signals are often characterized in terms of some properties in the Fourier-transformed domain. The class of finite-energy, Fourier-transformed bandlimited signals can easily be shown to be a specific realization of the abstract RKHS. The reproducing kernel is given by the familiar sinc function.

THEOREM 1. *Let H_1 be the class of $L_2(-\infty, \infty)$ functions such that their Fourier-transforms*

$$F(\omega) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} f(t)e^{-i\omega t}\, dt, \qquad f \in L_2(-A, A),$$

vanish almost everywhere outside of $(-\pi, \pi)$. Then H_1 is a reproducing kernel Hilbert space on $T_1 = (-\infty, \infty)$. The unique reproducing kernel

$K_1(s, t)$ is given by

$$K_1(s, t) = [\sin \pi(t - s)]/\pi(t - s). \tag{3}$$

Proof. From the inverse Fourier-transform, any $f \in H_1$ is given by

$$f(t) = \text{l.i.m.} \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{i\omega t} \, d\omega, \qquad F \in L_2(-\pi, \pi).$$

If $F \in L_2(-\pi, \pi)$, then $F \in L_1(-\pi, \pi)$. Schwarz inequality shows

$$|f(t)| \leqq \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} |F(\omega)|^2 \, d\omega \right]^{1/2} \left[\int_{-\pi}^{\pi} |e^{i\omega t}|^2 \, d\omega \right]^{1/2} < \infty,$$

for any $f \in H_1$ and any finite t. Thus, H_1 is a proper functional Hilbert space. Then H_1 is a specific realization of the abstract RKHS. The reproducing kernel given by Eq. (3) is obtained after applying the convolution theorem to the inverse Fourier-transform of the indicator function of $(-\pi, \pi)$.

In two-dimensional signal analysis, the signals are sometimes characterized in terms of some properties in the Bessel-transformed domain. The classes of finite-energy, Bessel-, sine-, and cosine-transformed band-limited signals can be shown to be specific realizations of the abstract RKHS in the same manner as that of Theorem 1.

COROLLARY 1. *Let H_2 be the class of $L_2(0, \infty)$ functions such that their Hankel-transforms of order ν, $\nu \geqq -\frac{1}{2}$,*

$$F(\omega) = \underset{A \to \infty}{\text{l.i.m.}} \int_0^A (\omega t)^{1/2} J_\nu(\omega t) f(t) \, dt, \qquad f \in L_2(0, A),$$

vanish almost everywhere outside of $(0, \pi)$. [The inverse Bessel-transform is given by

$$\left. f(t) = \text{l.i.m.} \int_0^{\pi} (\omega t)^{1/2} J_\nu(\omega t) F(\omega) \, d\omega, \qquad t \in (0, \infty) F \in L_2(0, \pi) \right].$$

Then H_2 is a reproducing kernel Hilbert space on $T_2 = (0, \infty)$. The unique reproducing kernel is given by

$$K_2(s, t) = \frac{\pi (ts)^{1/2}}{s^2 - t^2} \left(t J_\nu(s\pi) J_\nu'(t\pi) - s J_\nu(t\pi) J_\nu'(s\pi) \right).$$

COROLLARY 2. *Let H_3 be the class of $L_2(0, \infty)$ functions such that their* *sine-transforms*

$$F(\omega) = \underset{A\to\infty}{\text{l.i.m.}} \left(\frac{2}{\pi}\right)^{1/2} \int_0^A \sin \omega t f(t)\, dt, \qquad f \in L_2(0, \infty),$$

vanish almost everywhere outside of $(0, \pi)$. [*The inverse sine-transform is given by*

$$\left. f(t) = \text{l.i.m.} \left(\frac{2}{\pi}\right)^{1/2} \int_0^\pi \sin \omega t F(\omega)\, d\omega, t \in (0, \infty), F \in L_2(0, \pi) \right].$$

Then H_3 *is a reproducing kernel Hilbert space on* $T_3 = (0, \infty)$. *The unique reproducing kernel is given by*

$$K_3(s, t) = \frac{1}{\pi}\left(\frac{\sin \pi(t-s)}{(t-s)} - \frac{\sin \pi(t+s)}{(t+s)}\right).$$

COROLLARY 3. *Let* H_4 *be the class of* $L_2(0, \infty)$ *functions such that their cosine-transforms*

$$F(\omega) = \underset{A\to\infty}{\text{l.i.m.}} \left(\frac{2}{\pi}\right)^{1/2} \int_0^A \cos \omega t f(t)\, dt, \qquad f \in L_2(0, A),$$

vanish almost everywhere outside of $(0, \pi)$. [*The inverse cosine-transform is given by*

$$\left. f(t) = \text{l.i.m.} \left(\frac{2}{\pi}\right)^{1/2} \int_0^\pi \cos \omega t F(\omega)\, d\omega, t \in (0, \infty), F \in L_2(0, \pi) \right].$$

Then H_4 *is a reproducing kernel Hilbert space on* $T_4 = (0, \pi)$. *The unique reproducing kernel* $K_4(s, t)$ is given by

$$K_4(s, t) = \frac{1}{\pi}\left(\frac{\sin \pi(t-s)}{(t-s)} + \frac{\sin \pi(t+s)}{(t+s)}\right).$$

II. EXTREMUM PROBLEMS

In a given class of signals there are certain properties of the signals which are obtained from the solutions of extremum problems. If this class of signals is a RKHS, then the reproducing kernel plays an important role in these extremum problems. Suppose K is the reproducing kernel of a RKHS H, where H is a subspace of a Hilbert space Y. Then it turns out that the projection of any $y \in Y$ onto H is given by the inner product of y and K. (Aronszajn, 1950). For example, in H_1, the class of

Fourier bandlimited signals, maximum concentration properties of these signals over finite time intervals were obtained from self-adjoint compact integral equations where the kernel of the integral operator is given by the reproducing kernel $K_1(s, t)$. (Slepian and Pollak, 1961 and Landau and Pollak, 1961). In this section we shall consider two extremum problems in the abstract RKHS in which the sampling instants are specified. In these problems the reproducing kernel of the space again plays an important role

First consider a simple version of these two problems. Suppose t is a fixed point in the set T of the abstract RKHS H. What signal $f \in H$ with the specified value $f(t) = M$, where M is a real constant, has the smallest energy $\|f\|^2$? On the other hand, what signals $f \in H$ with energy $\|f\|^2 \leqq E$ have the maximum value for $f^2(t)$?

From Eq. (2) and the Schwarz inequality, it is clear that the solutions to both problems are the same. In the first case, the signal $f(s) = MK(s, t)/K(t, t)$ has the minimum energy $\|f\|^2 = M^2$ in the subspace of H with the constraint of $f(t) = M$. (Note that $K(t, t) = \|K\|^2 \neq 0$, unless K vanishes identically and thus H is void.) In the second case, $f(s) = \pm E^{1/2}K(s, t)/K(t, t)$ has the maximum value $f^2(t) = E$ in the subspace of H with the constraint of $\|f\|^2 \leqq E$.

Now consider the above two problems when there are n distinct, but arbitrarily-specified sampling instants, $t_i \in T$, $i = 1, \cdots, n$, and n real, finite, but arbitrarily-specified sampled values, M_i, $i = 1, \cdots, n$. Then the above two problems are generally not equivalent. Theorem 2 and Theorem 3 deal with these two problems.

In the first problem what signal $f \in H$ satisfying $f(t_i) = M_i$, $i = 1, \cdots, n$, (when t_i and M_i are specified), has the smallest energy $\|f\|^2$? The solution to a slightly extended version of this problem is given by Theorem 2 as the unique signal $f \in H$ which interpolates over a finite number of points and approximates any other specified signal $g \in H$ with minimum energy. When $g = 0$, Theorem 2 reduces to the first problem. The proof of Theorem 2 is based on the well-known Gram-Schmidt orthonormalization procedure so often used in interpolation theory. The proof of Theorem 2 is omitted since it is similar to Theorems 9.4.1 and 9.4.3 of Davis (1963).

THEOREM 2. *Consider the abstract reproducing kernel Hilbert space H and the reproducing kernel $K(s, t)$ defined on a set T. Let n be any finite positive integer, the sampling instants $\{t_1, \cdots, t_n\}$ be any set of finite distinct* *points in T, the sample values $\{M_1, \cdots, M_n\}$ be any set of real finite constants, and any $g \in H$. Denote H', the subspace of H under the interpolation constraints, to be*

$$H' = \{f \in H : f(t_i) = M_i, i = 1, \cdots, n\}.$$

Then

$$f_0(s) = g(s) + \sum_{i=1}^{n} d_i D_i(s)$$

is the unique element in H' that attains

$$\min_{f \in H'} \| f - g \|^2 = \| f_0 - g \|^2 = \sum_{i=1}^{n} d_i^2,$$

where for $i = 1, \cdots, n$,

$$d_i = \frac{1}{(G_{i-1} G_i)^{1/2}} \begin{vmatrix} K(t_1, t_1) & \cdots & K(t_n, t_1) \\ \vdots & & \\ K(t_1, t_{n-1}) & \cdots & K(t_n, t_{n-1}) \\ m_1 & \cdots & m_n \end{vmatrix},$$

$$m_i = M_i - g(t_i)$$

$$G_i = \det [K(t_j, t_k)]_{j,k=1,\cdots,i}, \qquad G_0 = 1$$

$$D_i(s) = \frac{1}{(G_{i-1} G_i)^{1/2}} \begin{vmatrix} K(t_1, t_1) & \cdots & K(t_n, t_1) \\ \vdots & & \\ K(t_1, t_{n-1}) & \cdots & K(t_n, t_{n-1}) \\ K(s, t_1) & \cdots & K(s, t_n) \end{vmatrix}.$$

In the second problem, for specified distinct sampling instants t_i, $i = 1, \cdots, n$, what signals $f \in H$ with energy $\| f \|^2 \leqq E$ yield the maximum value of $\sum_{i=1}^{n} f^2(t_i)$?

THEOREM 3. *Consider the abstract reproducing kernel Hilbert space H with the reproducing kernel $K(s, t)$ defined on a set T. Let n be any finite positive number and the sampling instants, $\{t_1 \cdots, t_n\}$, be any set of finite distinct points in T. Denote H'', the subspace of H under the energy constraint, to be*

$$H'' = \{f \in H : \| f \|^2 \leqq E\},$$

where E is a finite positive number. Then

$$f_0(s) = \pm\left[E/\lambda_n\left(\sum_{j=1}^{n}\theta_{n,j}^2\right)\right]^{1/2}\sum_{j=1}^{n}\theta_{n,j}K(s,t_j) \tag{4}$$

are elements in H'' that attain

$$\max_{f\in H''}\sum_{i=1}^{n}f^2(t_i) = \sum_{i=1}^{n}f_0^2(t_i) = \lambda_n E, \tag{5}$$

where λ_n is the largest eigenvalue and $\boldsymbol{\theta}_n$ is the corresponding eigenvector of the matrix equation

$$\begin{aligned}\mathbf{K}\boldsymbol{\theta} &= \lambda\boldsymbol{\theta}, \quad \textit{where}\\ \mathbf{K} &= [K(t_i, t_j)]_{i,j=1,\cdots,n},\\ \boldsymbol{\theta}_i^T &= [\theta_{i,1}, \cdots, \theta_{i,n}], \qquad i = 1, \cdots, n.\end{aligned} \tag{6}$$

Proof. From the defining property of the reproducing kernel and the Schwarz inequality,

$$\begin{aligned}\left[\sum_{i=1}^{n}f^2(t_i)\right]^2 &= \left[\sum_{i=1}^{n}f(t_i), (f(s), K(s,t_i))\right]^2\\ &= \left[\left(f(s), \sum_{i=1}^{n}f(t_i)K(s,t_i)\right)\right]^2\\ &\leqq \|f\|^2\left\|\sum_{i=1}^{n}f(t_i)K(s,t_i)\right\|^2\\ &= \|f\|^2\sum_{i=1}^{n}\sum_{j=1}^{n}f(t_i)f(t_j)K(t_i,t_j).\end{aligned} \tag{7}$$

In particular,

$$\frac{\sum_{i=1}^{n}f^2(t_i)}{\|f\|^2} \leqq \max_{\substack{f(t_i), i=1,\cdots,n\\ f\in H}} \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}f(t_i)f(t_j)K(t_i,t_j)}{\sum_{i=1}^{n}f^2(t_i)} = \lambda_n, \tag{8}$$

where λ_n is a positive constant. Let $f_1(t_i)$, $i = 1, \cdots, n$, be such that

$$\sum_{i=1}^{n}f_1^2(t_i) = A \tag{9}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}f_1(t_i)f_1(t_j)K(t_i,t_j) = A\lambda_n.$$

Then

$$\sum_{j=1}^{n} f_1(t_j)K(t_i, t_j) = \lambda_n f_1(t_i).$$

Let

$$h(s) = \sum_{j=1}^{n} f_1(t_j)K(s, t_j). \tag{10}$$

Then

$$\sum_{i=1}^{n} h^2(t_i) = A\lambda_n^2 \tag{11}$$

and

$$\| h \|^2 = A\lambda_n . \tag{12}$$

Thus, for $h(s)$ given by Eq. (10), the equality in Eq. (8) is attained. It is known that the maximum of the normalized quadratic form given by Eq. (8) is actually attained by the eigenvector, $\boldsymbol{\theta}_n$, corresponding to the largest eigenvalue, λ_n, of Eq. (6). Furthermore, the maximum of the normalized quadratic form is given by λ_n. (Courant and Hilbert, 1932). Thus, $f_1(t_i) = \theta_{n,i}$, $i = 1, \cdots, n$, and Eq. (4) follows from Eqs. (9), (10), and (11) while Eq. (5) follows from Eqs. (9), (10) and (12).

III. SAMPLING EXPANSIONS

In Section I, the classes of finite-energy, Fourier-, Hankel-, sine-, and cosine-transformed bandlimited signals were shown to be specific realizations of the abstract RKHS. For theoretical and practical reasons, sampling expansions may be of interest in these classes of signals. A class, Ω, of functions defined on a set T is said to possess a *sampling expansion* for a set of sampling instants $\{t_i \in T, i \in I\}$, if there exists a set of sampling functions, $\{\psi_i(s, t_i), i \in I\}$, such that

1. $\psi_i(s, t_i) \in \Omega, \qquad i \in I.$
2. $\psi_i(t_j, t_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$
3. For any $f \in \Omega$, there is a uniformly convergent expansion given by $f(s) = \sum_{i \in I} f(t_i)\psi_i(s, t_i), \qquad s \in T.$

Theorem 4 states one simple relationship between complete orthonormal expansions and sampling expansions in RKHS.

THEOREM 4. *Consider the abstract reproducing kernel Hilbert space H with the reproducing kernel $K(s, t)$ defined on a set T. Let $\{\phi_i(s, t_i), t_i \in T, i \in I\}$ be a complete orthonormal system in H. If there are non-zero real con-*

stants c_i, *such that*

$$\phi_i(s, t_i) = c_i K(s, t_i), \quad i \in I, \tag{13}$$

and

$$|K(t, t)| \leqq c < \infty, \qquad t \in T, \tag{14}$$

then the complete orthonormal expansion of any $f \in H$ *given by*

$$f(s) = \sum_{i \in I} a_i \phi_i(s, t_i), \qquad s \in T, \qquad a_i = (f, \phi_i), \tag{15}$$

is a sampling expansion.

Proof. From Eqs. (13) and (15),

$$a_i = (f, \phi_i) = c_i(f(s), K(s, t_i)) = c_i f(t_i).$$

Let $\psi_i(s, t_i) = c_i \phi_i(s, t_i)$. Then

$$\begin{aligned} c_i \phi_i(t_j, t_i) &= c_i(\phi_i(s, t_i), K(s, t_j)) \\ &= \frac{c_i}{c_j} (\phi_i(s, t_i), \phi_j(s, t_j)) = \begin{cases} 1, & j = i \\ 0, & j \neq i. \end{cases} \end{aligned}$$

Finally, in a RKHS, convergence in norm implies uniform convergence if $K(t, t)$ satisfies Eq. (14). (See Davis, 1963, Theorem 12.6.4).

In particular, the four sets of functions,

1. $\left\{\frac{\sin \pi(s - i)}{\pi(s - i)}, s \in T_1, -\infty < i < \infty\right\}$
2. $\left\{\frac{t_i(2s)^{1/2} J_\nu(\pi s)}{(t_i^2 - s^2)}, s \in T_2, \text{ where } \{\pi t_i\} \text{ are the positive zeros of } J_\nu, \nu \geqq -\frac{1}{2}, 1 \leqq i < \infty\right\}$
3. $\left\{\frac{2i}{(s + i)} \frac{\sin \pi(s - i)}{\pi(s - i)}, s \in T_3, 1 \leqq i < \infty\right\}$
4. $\left\{\frac{\sin \pi s}{\pi s}, \frac{2s}{(s + i)} \frac{\sin \pi(s - i)}{\pi(s - i)}, s \in T_4, 1 \leqq i < \infty\right\}$

are complete and orthonormal in H_j, where $j = 1, 2, 3, 4$, respectively. Since conditions required by Eqs. (13) and (14) in Theorem 4 are satisfied, we obtain the following four sampling expansions:

COROLLARY 4. *In RKHS* H_1, *any* $f \in H_1$, *possesses a sampling expansion given by*

$$f(s) = \sum_{i=-\infty}^{\infty} f(i) \frac{\sin \pi(s - i)}{\pi(s - i)}, \qquad -\infty < s < \infty. \tag{16}$$

COROLLARY 5. *In RKHS H_2, any $f \in H_2$, possesses a sampling expansion given by*

$$f(s) = \sum_{i=1}^{\infty} f(t_i) \frac{2(st_i)^{1/2}}{\pi J_{\nu+1}(\pi t_i)} \frac{J_\nu(\pi s)}{(t_i^2 - s^2)}, \qquad 0 < s < \infty, \tag{17}$$

where $\{\pi t_i\}$ are the positive zeros of J_ν, $\nu \geqq -\frac{1}{2}$.

COROLLARY 6. *In RKHS H_3, any $f \in H_3$, possesses a sampling expansion given by*

$$f(s) = \sum_{i=1}^{\infty} f(i) \frac{2i}{(s+i)} \frac{\sin \pi(s-i)}{\pi(s-i)}, \qquad 0 < s < \infty.$$

COROLLARY 7. *In RKHS H_4, any $f \in H_4$, possesses a sampling expansion given by*

$$f(s) = f(0) \frac{\sin \pi s}{\pi s} + \sum_{i=1}^{\infty} f(i) \frac{2s}{(s+i)} \frac{\sin \pi(s-i)}{\pi(s-i)}, \qquad 0 < s < \infty.$$

The sampling expansion given by Eq. (16) is generally known as the Shannon sampling theorem and was derived by Whittaker, E. (1912). The sampling expansion given by Eq. (17) was first discussed by Whittaker, J. (1935). For other references and discussions see Yao and Thomas (1965).

In practice, we have only a finite number of terms in any sampling expansion. Theorem 5 gives an upper bound on the truncation error when a finite number of terms are used in place of all the terms in the sampling expansion of Theorem 4.

THEOREM 5. *Consider the abstract RKHS H and the sampling expansion*

$$f(s) = \sum_{i \in I} f(t_i)\psi_i(s, t_i), \qquad s \in T, \qquad f \in H$$

of Theorem 4. Let I' be a proper subset of I with some finite number of integers. The truncation error, $E_{I'}(s)$, is defined by

$$E_{I'}(s) = \sum_{i \in (I-I')} f(t_i)\psi_i(s, t_i).$$

Then

$$|E_{I'}(s)| \leqq [E - \sum_{i \in I'} c_i^2 f^2(t_i)]^{1/2} [\sum_{i \in (I-I')} c_i^2 K^2(s, t_i)]^{1/2}, \qquad f \in H'' \tag{18}$$

is valid for any $f \in H''$, where $H'' = \{f \in H : \|f\|^2 \leqq E\}$. Furthermore, if $[E - \sum_{i \in I'} c_i^2 f^2(t_i)]$ is non-negative, there is a $f \in H''$ which attains the upper bound of $E_{I'}(s)$.

Proof. The hypercircle inequality of Golomb and Weinberger (1959) (See Davis, 1963, Theorem 9.4.7) states

$$| Lf - Lf_0 |^2 \leqq [E - \| f_0 \|^2][\sum_{i \in (I - I')} (L\phi_i)^2], \tag{19}$$

where L is a bounded linear functional in a Hilbert space S, $f \in S'' = \{g \in S: \| g \|^2 \leqq E\}$, f_0 is an element of smallest norm satisfying

$$(f_0, \phi_i) = b_i, \qquad i \in I',$$

$\{b_i, i \in I'\}$ are fixed constants, and $\{\phi_i, i \in I\}$ is a complete orthonormal system for S. Now, take $H = S$, $H'' = S''$, $b_i = c_i f(t_i)$,

$$f_0(s) = \sum_{i \in I'} f(t_i) c_i \phi_i(s, t_i) = \sum_{i \in I'} f(t_i) \psi_i(s, t_i),$$

$$f(s) = \sum_{i \in I} f(t_i) \psi_i(s, t_i), \qquad f \in H'',$$

and $L_s f = (f(u), K(u, s)) = f(s)$. Then Eq. (18) follows from Eq. (19). The attainment of the upper bound follows in the same manner as the attainment of the equality in the hypercircle inequality.

In the case of the Shannon sampling expansion, the result given in Theorem 5 can be used to calculate a simple upper truncation bound.

COROLLARY 8. *Consider the RKHS H_1 and the sampling expansion*

$$f(s) = \sum_{i=-\infty}^{\infty} f(i) \frac{\sin \pi(s - i)}{\pi(s - i)}, \qquad -\infty < s < \infty, \qquad f \in H_1.$$

Let $I' = \{i_0(s) - M \leqq i \leqq i_0(s) + N\}$, where $i_0(s)$ is the nearest integer to time s, and M and N are positive integers. The truncation error, $E_{M,N}(s)$, is given by

$$E_{M,N}(s) = \sum_{i=-\infty, i_0(s)-M-1}^{i_0(s)+N+1} f(i) \frac{\sin \pi(s - i)}{\pi(s - i)}, \qquad -\infty < s < \infty, \qquad f \in H_1.$$

For any $f \in H'' = \{f \in H_1 : \| f \|^2 \leqq E\}$, an upper bound of $E_{M,N}(s)$ is given by

$$| E_{M,N}(s) | < \begin{cases} \dfrac{E_0^{1/2}}{\pi} \left[\dfrac{1}{M} + \dfrac{2}{2N - 1}\right]^{1/2}, & s \in \left(i_0(s), i_0(s) + \dfrac{1}{2}\right] \\ \dfrac{E_0^{1/2}}{\pi} \left[\dfrac{2}{2M - 1} + \dfrac{1}{N}\right]^{1/2}, & s \in \left(i_0(s) - \dfrac{1}{2}, i_0(s)\right], \end{cases}$$

where

$$E_0 = \sum_{i=-\infty, i_0(s)+M-1}^{i_0(s)+N+1, \infty} f^2(i) < E < \infty .$$

Proof. From Theorem 5, in the case of the Shannon sampling expansion, $c_i = 1$ for all integers. Since $|\sin \pi(s - i)|$ is bounded by one and

$$\sum_{i=-\infty, i_0(s)-M-1}^{i_0(s)+N+1, \infty} \frac{1}{(s - i)^2} < \left[\frac{1}{M} + \frac{2}{2N - 1}\right], \qquad s \in \left(i_0(s), i_0(s) + \frac{1}{2}\right],$$

$$\sum_{i=-\infty, i_0 s)-M-1)}^{i_0(s)+N+1, \infty} \frac{1}{(s - i)^2} < \left[\frac{2}{2M - 1} + \frac{1}{N}\right], \quad s \in \left(i_0(s) - \frac{1}{2}, i_0(s)\right],$$

the upper bound of $E_{M,N}(s)$ follows immediately from Theorem 5.

Finally, we consider a sampling expansion problem in H_1 that relates various interpolation and minimum energy properties of Theorems 2, 3, 4, and 5. A finite Shannon sampling expansion given by

$$g(s) = C \sum_{i=1}^{n} f(t_i) \frac{\sin \pi(s - t_i)}{\pi(s - t_i)}, \tag{20}$$

where $C = 1$, $t_i = i$, $i = 1, \cdots, n$, $f \in H_1$, satisfies the following two properties:

$$\text{Interpolation Property} \quad g(t_i) = f(t_i), \; i = 1, \cdots, n, \tag{21}$$

$$\text{Minimum Energy Property} \quad \| g \|^2 = \min_{\substack{h \in h_1, \\ h(t_i) = f(t_i), \\ i=1,\cdots,n}} \| h \|^2. \tag{22}$$

In general, if the sampling instants, t_i, $i = 1, \cdots, n$, are distinct but arbitrary real points, the expansion given by Eq. (20) need not have the interpolation and minimum energy properties of Eqs. (21) and (22). A necessary and sufficient condition for the expansion given by Eq. (20) to satisfy the conditions of Eqs. (21), (22) is given in Theorem 6.

THEOREM 6. *A necessary and sufficient condition for a finite expansion given by Eq.* (20) *with arbitrary distinct real* $\{t_i, i = 1, \cdots, n\}$, *to satisfy the interpolation and minimum energy properties of Eqs.* (21), (22), *is that* $C = 1/\lambda_n$, $f(t_i) = \theta_{n,i}$ $i = 1, \cdots, n$, *where* λ_n *is the largest eigenvalue and* $\boldsymbol{\theta}_n$ *is the corresponding eigenvector of the matrix equation,*

$$\mathbf{K}\boldsymbol{\theta} = \lambda\boldsymbol{\theta},$$

$$\mathbf{K} = \left[\frac{\sin \pi(t_i - t_j)}{\pi(t_i - t_j)}\right]_{i,j=1,\cdots n}, \qquad \boldsymbol{\theta}^T = [\theta_1, \cdots, \theta_n].$$

Proof. Sufficiency: Let $f(t_i) = \theta_{n,i}$ and $C = 1/\lambda_n$. Then Eq. (19) becomes

$$g(t_j) = (1/\lambda_n) \sum_{i=1}^{n} \theta_{n,i} \frac{\sin \pi(t_j - t_i)}{\pi(t_j - t_i)} = \theta_{n,j} = f(t_j),$$

and Eq. (21) is satisfied. From Eq. (7), for any $h \in H_1$,

$$\left[\sum_{i=1}^{n} h^2(t_i)\right]^2 \leqq \| h \|^2 \left[\sum_{i=1}^{n} \sum_{j=1}^{n} h(t_i)h(t_j) \frac{\sin \pi(t_i - t_j)}{\pi(t_i - t_j)}\right].$$

Thus

$$\min_{\substack{h \in H_1, \\ h(t_i)=\theta_{n,i} \\ i=1,\cdots,n}} \| h \|^2 = (1/\lambda_n) \sum_{i=1}^{n} \theta_{n,i}^2. \tag{23}$$

By direct calculation

$$\| g \|^2 = (1/\lambda_n)^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_{n,i}\, \theta_{n,j} \frac{\sin \pi(t_i - t_j)}{\pi(t_i - t_j)} = (1/\lambda_n) \sum_{i=1}^{n} \theta_{n,i}^2. \tag{24}$$

Thus, Eqs. (23), (24) show that the property of Eq. (22) is satisfied by Eq. (20).

Necessity: Suppose Eq. (20) satisfies Eq. (21). That is,

$$f(t_j) = g(t_j) = C \sum_{i=1}^{n} f(t_i) \frac{\sin \pi(t_j - t_i)}{\pi(t_j - t_i)}, \qquad j = 1, \cdots, n.$$

Then

$$\mathbf{K}\hat{\mathbf{f}} = (1/C)\hat{\mathbf{f}}, \tag{25}$$

has eigenvalue $(1/C)$ and corresponding eigenvector $\hat{\mathbf{f}}$, where $\hat{\mathbf{f}}^T = [f(t_1), \cdots, f(t_n)]$. Suppose Eq. (20) satisfies Eq. (22). Then

$$\frac{\| g \|^2}{\sum_{i=1}^{n} f^2(t_i)} \min_{\substack{h \in H_1 \\ h(t_i)=f(t_i) \\ i=1,\cdots,n}} \frac{\| h \|^2}{\sum_{i=1}^{n} h^2(t_i)} = \frac{1}{\lambda_n}. \tag{26}$$

From Eqs. (23), (24), (25), and (26) then $C = 1/\lambda_n$ and $\hat{\mathbf{f}} = \boldsymbol{\theta}_n$.

In particular, if $t_i = i$, $i = 1, \cdots, n$, then $K(t_j - t_i) = \sin \pi(t_j - t_i)/\pi(t_j - t_i) = \delta_{ij}$ and $\lambda_n = 1 = C$. Then $\theta_{n,i}$, $i = 1, \cdots, n$, can take on any finite value. This conclusion clearly agrees with the previously known results.

IV. GENERALIZATIONS

Thus far, we have considered some of the more elementary aspects of reproducing kernel Hilbert spaces and bandlimited signals. It is clear that many generalizations are possible in several directions. Besides Fourier-, Hankel-, sine-, and cosine-transformed bandlimited signals, other unitary-transformed bandlimited signals can also be shown to be specific realizations of RKHS. Furthermore, if a class of bandlimited signals is a RKHS, then that class of bandpass signals is also a RKHS. The reproducing kernel in that case is obtained from the inverse transform of the indicator function of the support of the signals in the transformed domain.

If the number of sampling instants, n, is allowed to become infinite, then many of the above results are still valid. The proofs of some of these results then become considerably more involved. There are also analogous results in most of the above cases when constraints are imposed on the derivatives of the signals as well as the signals.

In conclusion, this paper has considered various new and known results in bandlimited signals and sampling expansions from the point of view of the RKHS. The actions of the reproducing kernel in extremum problems were emphasized. The RKHS approach simplified matters in some cases but offered a unified point of view in all cases.

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