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Numerical Analysis 2
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Exercise Sheet 2

1. Let $f \in C^n([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$, $n \in \mathbb{N}$. Show that the initial problem

$$y'(t) = f(t, y(t)), \quad y(0) = y_0$$

has for every initial value $y_0 \in \mathbb{R}^d$ a unique solution $y \in C^{n+1}([0,T_0] \times \mathbb{R}^d; \mathbb{R}^d)$ for some $T_0 \in (0,T]$.

2. We consider the differential equation

$$y'(t) = -y(t), \quad t > 0,$$

with the initial condition $y(0) = 1$. We solve the equation with the implicit Euler method for some fixed step size $h > 0$ and obtain the recurrence relation

$$y_{i+1} = \frac{1}{1+h} y_i + \varepsilon, \quad i \in \mathbb{N}_0,$$

with $y_0 = 1$. Here, $\varepsilon > 0$ shall model the rounding error. Show that the approximation error is bounded by

$$|y(ih) - y_i| \le \frac{1}{(1+h)^i} + \frac{1+h}{h}\,\varepsilon \quad \text{for all} \quad i \in \mathbb{N}_0.$$

How does this result change if we use the explicit Euler method instead?

3. Let us consider the recursive step

$$Y = y + hf(t, Y)$$

appearing in the implicit Euler method for some fixed step size $h > 0$ at some time $t \in (0,T)$. We assume that the function $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is continuously differentiable and fulfils the one-sided Lipschitz condition

$$\langle f(t,x) - f(t,z), x - z \rangle \le l \|x - z\|_2^2 \quad \text{for all} \quad x, z \in \mathbb{R}^d$$

for some constant $l \in \mathbb{R}$, so that the equation has a unique solution $Y \in \mathbb{R}^d$ if $hl < 1$. We additionally assume that f fulfils the Lipschitz condition

$$\|f(t,x) - f(t,z)\|_2 \le L \|x - z\|_2 \quad \text{for all} \quad x, z \in \mathbb{R}^d$$

with a Lipschitz constant $L > 0$.

(a) Show that the fixed point iteration

$$Y_{k+1} = y + hf(t, Y_k), \quad k \in \mathbb{N}_0,$$

converges for every initial value $Y_0 \in \mathbb{R}^d$ to the solution Y if $hL < 1$.

(b) Give an example of such a function f where the fixed point iteration does not converge for $hL = 1$.

4. Let $G \in \mathbb{R}^{d\times d}$ be a symmetric and positive definite matrix. We define the inner product $\langle x, z\rangle_G = x^{\mathrm{T}} Gz$ and the corresponding norm $\|x\|_G = \sqrt{\langle x, x\rangle_G}$ on $\mathbb{R}^d$. We further pick a function $f \in C^1([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ which fulfils the one-sided Lipschitz condition

$$\langle f(t,x) - f(t,z), x - z\rangle_G \le l\|x - z\|_G^2 \quad \text{for all} \quad t \in [0,T],\ x, z \in \mathbb{R}^d$$

with respect to this inner product for some Lipschitz constant $l < 0$.

We now construct with the implicit Euler method a sequence $(y_i)_{i=0}^{\infty} \subset \mathbb{R}^d$ approximating the solution $y \in C^2([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ of the initial value problem

$$y'(t) = f(t, y(t)), \quad y(0) = y_0$$

for an arbitrary $y_0 \in \mathbb{R}^d$:

$$y_{i+1} = y_i + hf(t_{i+1}, y_{i+1}), \quad t_i = ih,\ i \in \mathbb{N}_0,$$

for some step size $h > 0$. Show that we have the error estimate

$$\|y_i - y(t_i)\|_G \le \frac{h}{2|l|} \max_{t\in[0,T]} \|y''(t)\|_G, \quad i \in \mathbb{N}_0.$$

5. Let $(x_i)_{i=0}^{l}$ be a mesh on an interval $[a,b]$.

(a) Find a set $(p_j)_{j=0}^{l}$ of polynomials of degree l with the property

$$p_j(x_i) = \delta_{ij} \quad \text{for all} \quad i, j \in \{0, \dots, l\}.$$

(These polynomials are called Lagrange polynomials.)

(b) Write a program that calculates for an arbitrary function $f : [a,b] \to \mathbb{R}$ the interpolation polynomial p of degree l of f, which is defined by the property

$$p(x_i) = f(x_i) \quad \text{for all} \quad i \in \{0, \dots, l\}.$$

(c) We choose a uniform mesh $(x_i)_{i=0}^{l}$ on the interval $[-1,1]$ and pick as function

$$f : [-1,1] \to \mathbb{R}, \quad f(x) = \frac{1}{1 + 25x^2}.$$

Compare the shape of the interpolation polynomial of f with the approximation of f by its natural interpolating cubic spline for the values $l = 5, 10, 20$.

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