Exercise Sheet 6

1. We define for some $h \in (0, \frac{1}{2})$ the forward and backward difference quotient

$$D_h^+[u](x) = \frac{u(x+h) - u(x)}{h} \quad \text{and} \quad D_h^-[u](x) = \frac{u(x) - u(x-h)}{h}$$

of a function $u : [0,1] \to \mathbb{R}$ at a point $x \in [h, 1-h]$.

(a) Show that we have for every function $u \in C^2([0,1])$ at every point $x \in [h, 1-h]$ the estimates

$$|D_h^+[u](x) - u'(x)| \le \frac{h}{2}\|u''\|_\infty \quad \text{and} \quad |D_h^-[u](x) - u'(x)| \le \frac{h}{2}\|u''\|_\infty.$$

(b) Prove that for the second order difference quotient $D_h^2[u] = D_h^+[D_h^-[u]]$ of a function $u \in C^4([0,1])$, the estimate

$$|D_h^2[u] - u''(x)| \le \frac{h^2}{12}\|u''''\|_\infty$$

holds at every point $x \in [h, 1-h]$.

2. (a) Prove that the boundary value problem

$$-u''(x) + b(x)u'(x) + c(x)u(x) = f(x), \quad u(0) = 0, \quad u(1) = 0 \qquad (1)$$

has for all $b, c, f \in C([0,1])$ with $c(x) > 0$ for every $x \in [0,1]$ a unique solution $u \in C^2([0,1])$.
Hint: Show that the linear map

$$A : \mathbb{R}^2 \to \mathbb{R}^2, \quad A(\alpha, \gamma) = (\alpha, v(1))$$

where $v \in C^2([0,1])$ is the solution of the homogeneous problem

$$-v''(x) + b(x)v'(x) + c(x)v(x) = 0 \qquad (2)$$

with the initial conditions $v(0) = \alpha$ and $v'(0) = \gamma$ is bijective, i.e. the only solution of (2) with $v(0) = 0$ and $v(1) = 0$ is the function $v = 0$. Consider for this a point $\bar{x} \in (0,1)$ with $v(\bar{x}) = \max_{x \in [0,1]} v(x)$.

(b) Give an example of a boundary value problem of the form (1) (without the restriction $c(x) > 0$ for all $x \in [0,1]$) that has no solution.

(c) Does there also exist a boundary value problem of the form (1) (without the restriction $c(x) > 0$ for all $x \in [0,1]$) that has infinitely many solutions?

3. We consider the boundary value problem

$$- u''(x) + b(x)u'(x) + c(x)u(x) = f(x) \tag{3}$$

for $b, c, f \in C([0,1])$ with $c(x) \geq 0$ for every $x \in [0,1]$ with the inhomogeneous boundary conditions

$$u(0) = \alpha \quad \text{and} \quad u(1) = \beta, \tag{4}$$

for some given $\alpha, \beta \in \mathbb{R}$, for the function $u \in C^2([0,1])$.

(a) Show that this boundary value problem can be reduced to one of the form

$$-v''(x) + b(x)v'(x) + c(x)v(x) = \tilde{f}(x), \quad v(0) = 0, \quad v(1) = 0$$

for the function $v \in C^2([0,1])$ with some function $\tilde{f} \in C([0,1])$.

(b) Let $(x_i)_{i=0}^n$ be the uniform mesh on $[0,1]$ with step size $h = \frac{1}{n} \in (0, \frac{1}{2})$. We consider the finite difference method for the boundary value problem (3), (4) defined by the equations

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + b(x_i)\frac{u_{i+1} - u_{i-1}}{2h} + c(x_i)u_i = f(x_i), \quad i = 1, \ldots, n-1,$$

and the boundary conditions $u_0 = \alpha$ and $u_n = \beta$ for the approximation $(u_i)_{i=0}^n \subset \mathbb{R}$ of the solution u of the boundary value problem.

Determine the order of consistency of this finite difference method with respect to the maximum norm under the assumption that $u \in C^4([0,1])$.

4. Write a program that solves the boundary value problem (1) with the finite difference method obtained by using the central difference quotient $D_h[u]$ as approximation for u' and the second order difference quotient $D_h^2[u]$ as approximation for u'' on the uniform mesh $(x_i)_{i=0}^n$ on $[0,1]$ for given step size $h = \frac{1}{n} \in (0, \frac{1}{2})$.

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