Exercise Sheet 8

1. Let $u \in H_0^1((0,1))$ be the weak solution of the boundary value problem

$$\begin{aligned} -(\sigma u')'(x) + c(x)u(x) &= f(x) \quad \text{for almost all} \quad x \in (0,1), \\ u(0) = u(1) &= 0 \end{aligned}$$

for some bounded function $c : [0,1] \to \mathbb{R}$ with $c(x) \geq 0$ for all $x \in [0,1]$, some positive function $\sigma \in C^1([0,1])$, and some $f \in L^2((0,1))$.

   (a) Check that
$$w = \frac{1}{\sigma}(cu - f - \sigma' u') \in L^2((0,1)).$$

   (b) We define the function $W \in H^1((0,1))$ by
$$W(x) = \int_0^x w(t)\,\mathrm{d}t \quad \text{for all} \quad x \in (0,1)$$
and set $\omega = \int_0^1 W(x)\,\mathrm{d}x$. Prove the identity
$$\int_0^1 W(x)v(x)\,\mathrm{d}x = \int_0^1 (u'(x) + \omega)v(x)\,\mathrm{d}x \quad \text{for every } v \in L^2((0,1)). \tag{1}$$

   (c) Conclude that $u \in H^2((0,1)) \cap H_0^1((0,1))$.
   Hint: Use the identity (1) with the test function $v = W - u' - \omega$.

2. We consider the boundary value problem

$$\begin{aligned} -(\sigma u')'(x) &= x \quad \text{for all} \quad x \in (0,1), \\ u(0) = u(1) &= 0 \end{aligned} \tag{2}$$

where the function $\sigma : [0,1] \to \mathbb{R}$ is defined by

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in [0,\xi), \\ 2 & \text{if } x \in [\xi,1] \end{cases}$$

for some fixed value $\xi \in (0,1)$.

   (a) Show that the boundary value problem (2) cannot have a classical solution $u \in C^2((0,1))$.

(b) Show that the boundary value problem (2) admits a weak solution $u \in H_0^1((0,1))$ and that this solution u and the function $\sigma u'$ are continuous.

(c) Determine the weak solution u subject to the unknown value $u_0 = u(\xi) \in \mathbb{R}$ by solving the two boundary value problems

$$\begin{aligned} -u_1''(x) = x \quad &\text{for all} \quad x \in (0,\xi), \\ u_1(0) = 0 \quad &\text{and} \quad u_1(\xi) = u_0 \end{aligned}$$

for $u_1 \in C^2([0,\xi])$ and

$$\begin{aligned} -2u_2''(x) = x \quad &\text{for all} \quad x \in (\xi,1), \\ u_2(\xi) = u_0 \quad &\text{and} \quad u_2(1) = 0 \end{aligned}$$

for $u_2 \in C^2([\xi,1])$.

(d) Find the missing value u_0 by imposing the condition that the function $\sigma u'$ should be continuous at the point ξ.

(e) Verify that this function u is the weak solution of the boundary value problem (2).

3. Let Ω be an open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$ and $u \in H^k(\Omega)$, $k \in \mathbb{N}$. We pick a non-negative function $\omega \in C^\infty(\mathbb{R}^d)$ with

$$\omega(x) = 0 \quad \text{for} \quad \|x\| > 1 \quad \text{and} \quad \int_{\mathbb{R}^d} \omega(x)\,\mathrm{d}x = 1$$

and define for every $\rho > 0$ the functions $\omega_\rho \in C^\infty(\mathbb{R}^d)$,

$$\omega_\rho(x) = \rho^{-d}\omega(\tfrac{x}{\rho}), \quad x \in \mathbb{R}^d,$$

and $u_\rho \in C^\infty(\Omega)$,

$$u_\rho(x) = \int_\Omega \omega_\rho(x-y)u(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

(a) Show that we have for every multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ the identity

$$\partial^\alpha u_\rho(x) = \int_\Omega \omega_\rho(x-y)\partial^\alpha u(y)\,\mathrm{d}y$$

for all $x \in \Omega$ with $\operatorname{dist}(x, \mathbb{R}^d \setminus \Omega) > \rho$.

(b) Prove that

$$\lim_{\rho \to 0} \|u_\rho - u\|_{H^k(\Omega')} = 0$$

for every compact subset $\Omega' \subset \Omega$.
Hint: Use that every $f \in L^p(\mathbb{R}^d)$ fulfils

$$\sup_{\|z\|<\rho} \int_{\mathbb{R}^d} |f(x+z) - f(x)|^p\,\mathrm{d}x \to 0 \quad (\rho \to 0).$$

(c) Conclude that $C^\infty(\Omega) \cap H^k(\Omega)$ is dense in $H^k(\Omega)$.

4. Let us consider the boundary value problem

$$\begin{aligned} -\Delta u(x) &= 1 \quad \text{for all } x \in \Omega, \\ u(x) &= 0 \quad \text{for all } x \in \partial\Omega \end{aligned} \tag{3}$$

on the square $\Omega = (-1,1)^2$.

(a) Find analytically the solution $u \in C^2(\bar{\Omega})$ of this problem.

(b) Show that the triangulation obtained by dividing each square $(z_i, z_{i+1}) \times (z_j, z_{j+1})$ of the uniform grid on Ω, $z_i = \frac{i}{n}$, $i = 0, \ldots, n$, $n \in \mathbb{N}$, into the two triangles $T^+_{i,j}$ and $T^-_{i+1,j+1}$ with the vertices (z_i, z_j), (z_{i+1}, z_j), (z_i, z_{j+1}) and (z_{i+1}, z_{j+1}), (z_{i+1}, z_j), (z_i, z_{j+1}) gives a regular triangulation of Ω.

(c) Find the corresponding continuous functions $\Lambda_{(i,j)} : \Omega \to \mathbb{R}$ which are linear on every triangle $T^+_{(i,j)}$ and $T^-_{(i+1,j+1)}$, $i, j = 0, \ldots, n$, and are normalised by $\Lambda_{(i,j)}(z_{i'}, z_{j'}) = \delta_{i,i'}\delta_{j,j'}$.

(d) Calculate the stiffness matrix $A = (a_{(i,j),(k,\ell)})^n_{i,j,k,\ell=0} \in \mathbb{R}^{(n+1)^2} \times \mathbb{R}^{(n+1)^2}$ defined by

$$a_{(i,j),(k,\ell)} = \int_\Omega \left\langle \nabla\Lambda_{(i,j)}(x), \nabla\Lambda_{(k,\ell)}(x) \right\rangle \mathrm{d}x.$$

(e) Write a program that solves the boundary value problem (3) with the finite elements method.

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