Exercise Sheet 5

1. Consider the trapezoidal method

$$y_{i+1} = y_i + \frac{t_{i+1} - t_i}{2} \left[f(t_i, y_i) + f(t_{i+1}, y_{i+1}) \right],$$

to approximate the solution of the initial value problem

$$y'(t) = f(t, y(t)), \quad t \in [a, b], \quad y(t_0) = y_0,$$

in $n+1$ equidistant points in $[a, b]$. Solve the initial value problem

$$y'(t) = y(t) - t^2 + 1, \quad t \in [0, 1], \quad y(0) = \frac{1}{2}$$

for $n = 2$ using the implicit (backward) Euler method and the trapezoidal method.

2. Consider the following Runge-Kutta arrays

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \quad \text{and} \quad \begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

which define two second-order Runge-Kutta methods for approximating the solution of the initial value problem

$$y'(t) = -y(t), \quad t > 0, \quad y(0) = 1$$

For a given $h > 0$, find for both arrays the coefficients $C(h)$, such that the corresponding method takes the form

$$y_{i+1} = C(h)\, y_i$$

3. Consider the Runge-Kutta method with tableau

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Show that this method is A-stable.

4. Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad t \in [t_0, b],$$

and it's perturbation

$$y_\epsilon'(t) = f(t, y_\epsilon(t)), \quad y_\epsilon(t_0) = y_0 + \epsilon, \ \epsilon > 0, \quad t \in [t_0, b],$$

An initial value problem is considered to be well-conditioned if

$$\|y_\epsilon - y\|_\infty = \max_{0 \le t \le b} |y_\epsilon(t) - y(t)| \le c\,\epsilon,$$

for some $c > 0$ independent of ϵ. Consider the problems

(a)
$$y'(t) = \lambda(y(t) - 1), \quad \lambda \in \mathbb{R}, \quad t \in [0, b],$$
with general solution
$$y(t) = 1 + c_a\, e^{\lambda t}, \quad c_a \in \mathbb{R}.$$

(b)
$$y'(t) = -y^2(t), \quad t \in [0, b],$$
with general solution
$$y(t) = \frac{1}{t - c_b}, \quad c_b \in \mathbb{R}.$$

Set as initial condition $y(0) = 1$ in both of the problems and characterize them with respect to stability.

5. Implement in Matlab the Euler method and the trapezoidal method (ex. 1) to approximate the exact solution $y(t) = e^{t - t^2/2}$ of the initial value problem

$$y'(t) = (1 - t)y(t), \quad y(0) = 1, \quad t \in [0, 2],$$

for $h := t_{i+1} - t_i = 0.5$, 0.2 and 0.1.

6. The fourth-order Runge-Kutta method is given by

$$\begin{aligned}
y_0 &= y(a),\\
k_0 &= hf(t_i, y_i),\\
k_1 &= hf(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_0),\\
k_2 &= hf(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_1),\\
k_3 &= hf(t_{i+1}, y_i + k_2),\\
y_{i+1} &= y_i + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3), \quad i = 0, ..., n-1.
\end{aligned}$$

Implement in Matlab the above method to approximate the solution of the initial value problem

$$y'(t) = -\frac{y(t)}{1 + t}, \quad t \in [0, 1], \quad y(0) = 1, \quad \text{for} \quad h = 0.005.$$