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Exercise Sheet 2

1. Consider the system
$$Ax = b, \tag{1}$$
where A is a symmetric positive definite matrix. Let M be a symmetric positive definite matrix, which can be written as $M = R^T R$, using the Cholesky decomposition, where R is an upper triangular matrix. Show that

   (a) the matrix $\tilde{A} = (R^{-1})^T A R^{-1}$ is positive definite.

   (b) the system (1) is equivalent to the system
$$\begin{aligned} \tilde{A}y &= \tilde{b}, \\ y &= Rx, \end{aligned} \tag{2}$$
   where $\tilde{b} = (R^{-1})^T b$.

   (c) if x_0 and y_0 are two approximated solutions of the systems (1) and (2) respectively, then
$$\|y - y_0\|_{\tilde{A}} = \|x - x_0\|_A ,$$
   where the norms are the energy norms induced by $\tilde{A}$ and A, respectively.

2. Consider the constrained optimization problem
$$\min \quad f(x) \quad \text{subject to} \quad c(x) = 0,$$
where $f : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}^m$, are three times continuously differentiable on $\mathbb{R}^n$ and $m \leq n$. The problem is equivalent to search for solutions $(x^*, \lambda^*) \in \mathbb{R}^n \times \mathbb{R}^m$ of the system
$$\begin{aligned} \nabla_x L(x^*, \lambda^*) &= 0, \\ c(x^*) &= 0, \end{aligned}$$

where $L : \mathbb{R}^{n+m} \to \mathbb{R}$ is the Lagrangian function, given by

$$L(x, \lambda) = f(x) + \lambda^T c(x).$$

For $a, b > 0$ we define the penalty function

$$P(x, \lambda; b, a) = L(x, \lambda) + \tfrac{1}{2} \nabla L(x, \lambda)^T K(b, a) \nabla L(x, \lambda),$$

where

$$K(b, a) = \begin{bmatrix} aI & 0 \\ 0 & bI \end{bmatrix} \quad \text{and} \quad \nabla L(x, \lambda) = \begin{bmatrix} \nabla_x L(x, \lambda) \\ \nabla_\lambda L(x, \lambda) \end{bmatrix}$$

and I is the identity matrix of appropriate dimensions. Let $z \in \mathbb{R}^n$ such that

$$\nabla c(x^*)^T z = 0 \quad \text{and} \quad z^T \nabla^2_{xx} L(x^*, \lambda^*) z < 0.$$

Show that there exists $\bar{a} > 0$ such that, for all $a \in (0, \bar{a})$ and $b > 0$, the pair (x^*, λ^*) is not an unconstrained local minimum of $P(\cdot, \cdot; b, a)$.

3. Consider the equality constrained quadratic programming (QP) problem

$$\begin{aligned} \min \ & f(\mathbf{x}) = \tfrac{1}{2} \mathbf{x}^T A \mathbf{x} + \mathbf{e}_1^T \mathbf{x} \\ \text{subject to} \ & x_1 + 2x_2 + x_3 = 4, \end{aligned} \tag{3}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Eliminate the variable x_1 in order to express the resulting function in the form

$$\tfrac{1}{2} \mathbf{y}^T B \mathbf{y} + v^T \mathbf{y} + c,$$

where $\mathbf{y}^T = (x_2 \ x_3)$, B is a constant symmetric matrix, v is a constant vector and $c \in \mathbb{R}$.

(b) Find the solution $\mathbf{x}^*$ of the QP problem.

(c) Find the Lagrange multiplier λ^* of the equality constraint.