ON THE X-RAY TRANSFORM OF PLANAR SYMMETRIC 2-TENSORS

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ABSTRACT. In this paper we study the attenuated X-ray transform of 2-tensors supported in convex bounded subsets with sufficiently smooth boundary in the Euclidean plane. We characterize its range and reconstruct all possible 2-tensors yielding identical X-ray data. The characterization is in terms of a Hilbert-transform associated with A-analytic maps in the sense of Bukhgeim.

1. INTRODUCTION

This paper concerns the range characterization of the attenuated X-ray transform of symmetric 2-tensors in the plane. Range characterization of the non-attenuated X-ray transform of functions (0-tensors) in the Euclidean space has been long known [10, 11, 19], whereas in the case of a constant attenuation some range conditions can be inferred from [17, 1, 2]. For a varying attenuation the two dimensional case has been particularly interesting with inversion formulas requiring new analytical tools: the theory of A-analytic maps originally employed in [3], and ideas from inverse scattering in [24]. Constraints on the range for the two dimensional X-ray transform of functions were given in [25, 4], and a range characterization based on Bukhgeim's theory of A-analytic maps was given in [30].

Inversion of the X-ray transform of higher order tensors has been formulated directly in the setting of Riemmanian manifolds with boundary [32]. The case of 2-tensors appears in the linearization of the boundary rigidity problem. It is easy to see that injectivity can hold only in some restricted class: e.g., the class of solenoidal tensors. For two dimensional simple manifolds with boundary, injectivity with in the solenoidal tensor fields has been establish fairly recent: in the non-attenuated case for 0- and 1-tensors we mention the breakthrough result in [29], and in the attenuated case in [34]; see also [13] for a more general weighted transform. Inversion for the attenuated X-ray transform for solenoidal tensors of rank two and higher

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can be found in [27], with a range characterization in [28]. In the Euclidean case we mention an earlier inversion of the attenuated X-ray transform of solenoidal tensors in [16]; however this work does not address range characterization.

Different from the recent characterization in terms of the scattering relation in [28], in this paper the range conditions are in terms of the Hilbert-transform for A-analytic maps introduced in [30, 31]. Our characterization can be understood as an explicit description of the scattering relation in [26, 27, 28] particularized to the Euclidean setting. In the sufficiency part we reconstruct all possible 2-tensors yielding identical X-ray data; see (30) for the non-attenuated case and (82) for the attenuated case.

For a real symmetric 2-tensor $\mathbf{F} \in L^1(\mathbb{R}^2; \mathbb{R}^{2\times 2})$,

$$\mathbf{F}(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{12}(x) & f_{22}(x) \end{pmatrix}, \quad x \in \mathbb{R}^2, \tag{1}$$

and a real valued function $a \in L^1(\mathbb{R}^2)$, the *a-attenuated X-ray transform of* $\mathbf{F}$ is defined by

$$X_a\mathbf{F}(x,\theta) := \int_{-\infty}^{\infty} \langle \mathbf{F}(x+t\theta)\,\theta, \theta\rangle \exp\left\{ -\int_t^{\infty} a(x+s\theta)ds \right\} dt, \tag{2}$$

where θ is a direction in the unit sphere $\mathbf{S}^1$, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^2$. For the non attenuated case $a \equiv 0$ we use the notation $X\mathbf{F}$.

In this paper, we consider $\mathbf{F}$ be defined on a strongly convex bounded set $\Omega \subset \mathbb{R}^2$ with vanishing trace at the boundary Γ; further regularity and the order of vanishing will be specified in the theorems. In particular, in the attenuated case we assume that Γ is $C^{2,\alpha}, \alpha > \frac{1}{2}$ smooth. We also assume $a > 0$ in $\overline{\Omega}$.

For any $(x,\theta) \in \overline{\Omega} \times \mathbf{S}^1$ let $\tau(x,\theta)$ be length of the chord in the direction of θ passing through x. Let also consider the incoming $(-)$, respectively outgoing $(+)$ submanifolds of the unit bundle restricted to the boundary

$$\Gamma_\pm := \{(x,\theta) \in \Gamma \times \mathbf{S}^1 : \pm\theta \cdot n(x) > 0\}, \tag{3}$$

and the variety

$$\Gamma_0 := \{(x,\theta) \in \Gamma \times \mathbf{S}^1 : \theta \cdot n(x) = 0\}, \tag{4}$$

where $n(x)$ denotes outer normal.

The a-attenuated X-ray transform of $\mathbf{F}$ is realized as a function on $\mathit{\Gamma}_+$ by

$$X_a\mathbf{F}(x,\theta) = \int_{-\tau(x,\theta)}^{0} \langle \mathbf{F}(x+t\theta)\,\theta, \theta\rangle\, e^{-\int_t^0 a(x+s\theta)ds}\, dt, \ (x,\theta) \in \mathit{\Gamma}_+. \tag{5}$$

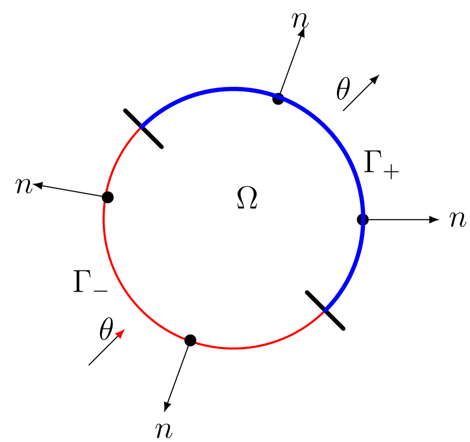

FIGURE 1. Definition of $\varGamma_\pm$

We approach the range characterization through its connection with the transport model as follows: The boundary value problem

$$\theta \cdot \nabla u(x,\theta) + a(x)u(x,\theta) = \langle \mathbf{F}(x)\theta, \theta\rangle \quad (x,\theta) \in \Omega \times \mathbf{S}^1, \tag{6}$$

$$u|_{\varGamma_-} = 0 \tag{7}$$

has a unique solution in $\Omega \times \mathbf{S}^1$ and

$$u|_{\varGamma_+}(x,\theta) = X_a\mathbf{F}(x,\theta), \quad (x,\theta) \in \varGamma_+. \tag{8}$$

The X-ray transform of 2-tensors occurs in the linearization of the boundary rigidity problem [32]: For $\epsilon > 0$ small, let

$$g^\epsilon(x) := \mathbf{I} + \epsilon\mathbf{F}(x) + o(\epsilon), \; x \in \Omega,$$

be a family of metrics perturbations from the Euclidean, where $\mathbf{I}$ is the identity matrix and $\mathbf{F}$ is as in (1). For an arbitrary pair of boundary points $x, y \in \varGamma$ let $d_\epsilon(x,y)$ denote their distance in the metric g^ϵ. The boundary rigidity problem asks for the recovery of the metric g^ϵ from knowledge of $d_\epsilon(x,y)$ for all $x, y \in \varGamma$. In the linearized case one seeks to recover $\mathbf{F}(x)$ from $\frac{d}{d\epsilon}\big|_{\epsilon=0} d^2_\epsilon(x,y)$. Taking into account the length minimizing property of geodesic one can show that

$$\frac{1}{|x-y|}\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} d^2_\epsilon(x,y) = \int_{-|x-y|}^{0} \langle \mathbf{F}(x+t\theta)\theta,\theta\rangle dt = X\mathbf{F}(x,\theta),$$

where $\theta := \dfrac{x-y}{|x-y|} \in \mathbf{S}^1$.

2. Preliminaries

In this section we briefly introduce the properties of Bukhgeim's A-analytic maps [7] needed later.

For $z = x_1 + ix_2$, we consider the Cauchy-Riemann operators

$$\overline{\partial} = (\partial_{x_1} + i\partial_{x_2})/2, \quad \partial = (\partial_{x_1} - i\partial_{x_2})/2. \tag{9}$$

Let $l_\infty(, l_1)$ be the space of bounded (, respectively summable) sequences, $\mathcal{L} : l_\infty \to l_\infty$ be the left shift

$$\mathcal{L}\langle u_{-1}, u_{-2}, ...\rangle = \langle u_{-2}, u_{-3}, u_{-4}, ...\rangle.$$

Definition 2.1. *A sequence valued map*

$$z \mapsto \mathbf{u}(z) := \langle u_{-1}(z), u_{-2}(z), u_{-3}(z), ...\rangle$$

is called $\mathcal{L}$-analytic*, if* $\mathbf{u} \in C(\overline{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty)$ *and*

$$\overline{\partial}\mathbf{u}(z) + \mathcal{L}\partial\mathbf{u}(z) = 0, \quad z \in \Omega. \tag{10}$$

For $0 < \alpha < 1$ and $k = 1, 2$, we recall the Banach spaces in [30]:

$$l^{1,k}_\infty(\mathit{\Gamma}) := \left\{ \mathbf{u} = \langle u_{-1}, u_{-2}, ...\rangle : \sup_{\zeta\in\mathit{\Gamma}} \sum_{j=1}^{\infty} j^k |u_{-j}(\zeta)| < \infty \right\}, \tag{11}$$

$$C^\alpha(\mathit{\Gamma}; l_1) := \left\{ \mathbf{u} : \sup_{\xi\in\mathit{\Gamma}} \|\mathbf{u}(\xi)\|_{l_1} + \sup_{\substack{\xi,\eta\in\mathit{\Gamma} \\ \xi\neq\eta}} \frac{\|\mathbf{u}(\xi) - \mathbf{u}(\eta)\|_{l_1}}{|\xi - \eta|^\alpha} < \infty \right\}. \tag{12}$$

By replacing $\mathit{\Gamma}$ with $\overline{\Omega}$ and l_1 with l_∞ in (12) we similarly define $C^\alpha(\overline{\Omega}; l_1)$, respectively, $C^\alpha(\overline{\Omega}; l_\infty)$.

At the heart of the theory of A-analytic maps lies a Cauchy-like integral formula introduced by Bukhgeim in [7]. The explicit variant (13) appeared first in Finch [8]. The formula below is restated in terms of $\mathcal{L}$-analytic maps as in [31].

Theorem 2.1. [31, Theorem 2.1] *For some* $\mathbf{g} = \langle g_{-1}, g_{-2}, g_{-3}, ...\rangle \in l^{1,1}_\infty(\mathit{\Gamma}) \cap C^\alpha(\mathit{\Gamma}; l_1)$ *define the Bukhgeim-Cauchy operator* $\mathcal{B}$ *acting on* $\mathbf{g}$*,*

$$\Omega \ni z \mapsto \langle(\mathcal{B}\mathbf{g})_{-1}(z), (\mathcal{B}\mathbf{g})_{-2}(z), (\mathcal{B}\mathbf{g})_{-3}(z), ...\rangle,$$

by

$$\begin{aligned}(\mathcal{B}\mathbf{g})_{-n}(z) := &\frac{1}{2\pi i}\sum_{j=0}^{\infty}\int_{\mathit{\Gamma}} \frac{g_{-n-j}(\zeta)\overline{(\zeta - z)}^j}{(\zeta - z)^{j+1}} d\zeta \\ &- \frac{1}{2\pi i}\sum_{j=1}^{\infty}\int_{\mathit{\Gamma}} \frac{g_{-n-j}(\zeta)\overline{(\zeta - z)}^{j-1}}{(\zeta - z)^j} d\overline{\zeta}, \; n = 1, 2, 3, ...\end{aligned} \tag{13}$$

Then $\mathcal{B}\mathbf{g} \in C^{1,\alpha}(\Omega; l_\infty) \cap C(\overline{\Omega}; l_\infty)$ *and it is also* $\mathcal{L}$*-analytic.*

For our purposes further regularity in $\mathcal{B}\mathbf{g}$ will be required. Such smoothness is obtained by increasing the assumptions on the rate of decay of the terms in $\mathbf{g}$ as explicit below. For $0 < \alpha < 1$, let us recall the Banach space Y_α in [30]:

$$Y_\alpha = \left\{ \mathbf{g} \in l^{1,2}_\infty(\Gamma) : \sup_{\substack{\xi,\mu\in\Gamma \\ \xi\neq\mu}} \sum_{j=1}^{\infty} j \frac{|g_{-j}(\xi) - g_{-j}(\mu)|}{|\xi-\mu|^\alpha} < \infty \right\}. \tag{14}$$

Proposition 2.1. [31, Proposition 2.1] *If* $\mathbf{g} \in Y_\alpha$, $\alpha > 1/2$, *then*

$$\mathcal{B}\mathbf{g} \in C^{1,\alpha}(\Omega; l_1) \cap C^\alpha(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty). \tag{15}$$

The Hilbert transform associated with boundary of $\mathcal{L}$-analytic maps is defined below.

Definition 2.2. *For* $\mathbf{g} = \langle g_{-1}, g_{-2}, g_{-3}, \ldots\rangle \in l^{1,1}_\infty(\varGamma) \cap C^\alpha(\varGamma; l_1)$, *we define the Hilbert transform* $\mathcal{H}\mathbf{g}$ *componentwise for* $n \geq 1$ *by*

$$\begin{aligned}(\mathcal{H}\mathbf{g})_{-n}(\xi) &= \frac{1}{\pi}\int_\varGamma \frac{g_{-n}(\zeta)}{\zeta-\xi} d\zeta \\ &\quad + \frac{1}{\pi}\int_\varGamma \left\{ \frac{d\zeta}{\zeta-\xi} - \frac{d\overline{\zeta}}{\overline{\zeta-\xi}} \right\} \sum_{j=1}^{\infty} g_{-n-j}(\zeta) \left(\frac{\overline{\zeta-\xi}}{\zeta-\xi}\right)^j, \; \xi \in \varGamma.\end{aligned} \tag{16}$$

The following result justifies the name of the transform $\mathcal{H}$. For its proof we refer to [30, Theorem 3.2].

Theorem 2.2. *For* $0 < \alpha < 1$, *let* $\mathbf{g} \in l^{1,1}_\infty(\Gamma) \cap C^\alpha(\Gamma; l_1)$. *For* $\mathbf{g}$ *to be boundary value of an* $\mathcal{L}$*-analytic function it is necessary and sufficient that*

$$(I + i\mathcal{H})\mathbf{g} = \mathbf{0}, \tag{17}$$

where $\mathcal{H}$ *is as in* (16).

3. THE NON-ATTENUATED CASE

In this section we assume $a \equiv 0$. We establish necessary and sufficient conditions for a sufficiently smooth function on $\varGamma \times \mathbf{S}^1$ to be the X-ray data of some sufficiently smooth real valued symmetric 2-tensor $\mathbf{F}$. For $\theta = (\cos\varphi, \sin\varphi) \in \mathbf{S}^1$, a calculation shows that

$$\langle \mathbf{F}(x)\theta, \theta\rangle = f_0(x) + \overline{f_2(x)} e^{2i\varphi} + f_2(x) e^{-2i\varphi}, \tag{18}$$

where

$$f_0(x) = \frac{f_{11}(x) + f_{22}(x)}{2}, \text{ and } f_2(x) = \frac{f_{11}(x) - f_{22}(x)}{4} + i\frac{f_{12}(x)}{2}. \tag{19}$$

The transport equation in (6) becomes

$$\theta \cdot \nabla u(x,\theta) = f_0(x) + \overline{f_2(x)} e^{2i\varphi} + f_2(x) e^{-2i\varphi}, \quad x \in \Omega. \tag{20}$$

For $z = x_1 + ix_2 \in \Omega$, we consider the Fourier expansions of $u(z,\cdot)$ in the angular variable $\theta = (\cos\varphi, \sin\varphi)$:

$$u(z,\theta) = \sum_{-\infty}^{\infty} u_n(z) e^{in\varphi}.$$

Since u is real valued its Fourier modes occur in conjugates,

$$u_{-n}(z) = \overline{u_n(z)}, \quad n \geq 0,\ z \in \Omega.$$

With the Cauchy-Riemann operators defined in (9) the advection operator becomes

$$\theta \cdot \nabla = e^{-i\varphi}\overline{\partial} + e^{i\varphi}\partial.$$

Provided appropriate convergence of the series (given by smoothness in the angular variable) we see that if u solves (20) then its Fourier modes solve the system

$$\overline{\partial} u_1(z) + \partial u_{-1}(z) = f_0(z), \tag{21}$$

$$\overline{\partial} u_{-1}(z) + \partial u_{-3}(z) = f_2(z), \tag{22}$$

$$\overline{\partial} u_{2n}(z) + \partial u_{2n-2}(z) = 0, \quad n \leq 0, \tag{23}$$

$$\overline{\partial} u_{2n-1}(z) + \partial u_{2n-3}(z) = 0, \quad n \leq -1, \tag{24}$$

The range characterization is given in terms of the trace

$$g := u|_{\varGamma \times \mathbf{S}^1} = \begin{cases} X\mathbf{F}(x,\theta), & (x,\theta) \in \varGamma_+, \\ 0, & (x,\theta) \in \varGamma_- \cup \varGamma_0. \end{cases} \tag{25}$$

More precisely, in terms of its Fourier modes in the angular variables:

$$g(\zeta,\theta) = \sum_{-\infty}^{\infty} g_n(\zeta) e^{in\varphi}, \quad \zeta \in \varGamma. \tag{26}$$

Since the trace g is also real valued, its Fourier modes will satisfy

$$g_{-n}(\zeta) = \overline{g_n(\zeta)}, \quad n \geq 0,\ \zeta \in \varGamma. \tag{27}$$

From the negative even modes, we built the sequence

$$\mathbf{g}^{even} := \langle g_0, g_{-2}, g_{-4}, ...\rangle. \tag{28}$$

From the negative odd modes starting from mode -3, we built the sequence

$$\mathbf{g}^{odd} := \langle g_{-3}, g_{-5}, g_{-7}, ...\rangle. \tag{29}$$

Next we characterize the data g in terms of the Hilbert Transform $\mathcal{H}$ in (16). We will construct simultaneously the right hand side of the transport equation (20) and the solution u whose trace matches the boundary data g. Construction of u is via its Fourier modes. We first construct the negative modes and then the positive modes are constructed by conjugation. Except for negative one mode u_{-1} all non-positive modes are defined by Bukhgeim-Cauchy integral formula in (13) using boundary data. Other than having the trace g_{-1} on the boundary u_{-1} is unconstrained. It is chosen arbitrarily from the class of functions

$$\Psi_g := \left\{ \psi \in C^1(\overline{\Omega}; \mathbb{C}) : \psi|_{\varGamma} = g_{-1} \right\}. \tag{30}$$

Theorem 3.1 (Range characterization in the non-attenuated case)**.** *Let* $\alpha > 1/2$.

(i) Let $\mathbf{F} \in C_0^{1,\alpha}(\Omega; \mathbb{R}^{2\times 2})$. *For* $g := \begin{cases} X\mathbf{F}(x,\theta), & (x,\theta) \in \varGamma_+, \\ 0, & (x,\theta) \in \varGamma_- \cup \varGamma_0, \end{cases}$ *consider the corresponding sequences* $\mathbf{g}^{even}$ *as in* (28) *and* $\mathbf{g}^{odd}$ *as in* (29). *Then* $\mathbf{g}^{even}, \mathbf{g}^{odd} \in l^{1,1}_{\infty}(\Gamma) \cap C^{\alpha}(\Gamma; l_1)$ *satisfy*

$$[I + i\mathcal{H}]\mathbf{g}^{even} = \mathbf{0}, \tag{31}$$

$$[I + i\mathcal{H}]\mathbf{g}^{odd} = \mathbf{0}, \tag{32}$$

where the operator $\mathcal{H}$ *is the Hilbert transform in* (16).

(ii) Let $g \in C^{\alpha}(\varGamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C(\varGamma; C^{2,\alpha}(\mathbf{S}^1))$ *be real valued with* $g|_{\varGamma_- \cup \varGamma_0} = 0$. *If the corresponding sequence* $\mathbf{g}^{even}, \mathbf{g}^{odd} \in Y_\alpha$ *satisfies* (31) *and* (32)*, then there exists a real valued symmetric 2-tensor* $\mathbf{F} \in C(\Omega; \mathbb{R}^{2\times 2})$*, such that* $g|_{\varGamma_+} = X\mathbf{F}$. *Moreover for each* $\psi \in \Psi_g$ *in* (30)*, there is a unique real valued symmetric 2-tensor* $\mathbf{F}_\psi$ *such that* $g|_{\varGamma_+} = X\mathbf{F}_\psi$.

Proof. (i) **Necessity**

Let $\mathbf{F} \in C_0^{1,\alpha}(\Omega; \mathbb{R}^{2\times 2})$. Since $\mathbf{F}$ is compactly supported inside Ω, for any point at the boundary there is a cone of lines which do not meet the support. Thus $g \equiv 0$ in the neighborhood of the variety $\varGamma_0$ which yields $g \in C^{1,\alpha}(\varGamma \times \mathbf{S}^1)$. Moreover, g is the trace on $\varGamma \times \mathbf{S}^1$ of a solution $u \in C^{1,\alpha}(\overline{\Omega} \times \mathbf{S}^1)$ of the transport equation (20). By [30, Proposition 4.1] $\mathbf{g}^{even}, \mathbf{g}^{odd} \in l^{1,1}_{\infty}(\Gamma) \cap C^{\alpha}(\Gamma; l_1)$.

If u solves (20) then its Fourier modes satisfy (21), (22), (23) and (24). Since the negative even Fourier modes u_{2n} of u satisfies the system (23) for $n \leq 0$, then

$$z \mapsto \mathbf{u}^{even}(z) := \langle u_0(z), u_{-2}(z), u_{-4}(z), u_{-6}(z), \cdots \rangle$$

is $\mathcal{L}$-analytic in Ω and the necessity part in Theorem 2.2 yields (31).

The equation (24) for negative odd Fourier modes u_{2n-1} starting from mode -3 yield that the sequence valued map

$$z \mapsto \mathbf{u}^{odd}(z) := \langle u_{-3}(z), u_{-5}(z), u_{-7}(z), \cdots \rangle$$

is $\mathcal{L}$-analytic in Ω and the necessity part in Theorem 2.2 yields (32).

(ii) **Sufficiency**

To prove the sufficiency we will construct a real valued symmetric 2-tensor $\mathbf{F}$ in Ω and a real valued function $u \in C^1(\Omega \times \mathbf{S}^1) \cap C(\overline{\Omega} \times \mathbf{S}^1)$ such that $u|_{\varGamma \times \mathbf{S}^1} = g$ and u solves (20) in Ω. The construction of such u is in terms of its Fourier modes in the angular variable and it is done in several steps.

Step 1: The construction of negative even modes u_{2n} for $n \leq 0$.

Let $g \in C^\alpha(\varGamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C(\varGamma; C^{2,\alpha}(\mathbf{S}^1))$ be real valued with $g|_{\varGamma_- \cup \varGamma_0} = 0$. Let the corresponding sequences $\mathbf{g}^{even}$ satisfying (31) and $\mathbf{g}^{odd}$ satisfying (32). By [30, Proposition 4.1(ii)] $\mathbf{g}^{even}, \mathbf{g}^{odd} \in Y_\alpha$. Use the Bukhgeim-Cauchy Integral formula (13) to construct the negative even Fourier modes:

$$\langle u_0(z), u_{-2}(z), u_{-4}(z), u_{-6}(z), ...\rangle := \mathcal{B}\mathbf{g}^{even}(z), \quad z \in \Omega. \tag{33}$$

By Theorem 2.1, the sequence valued map

$$z \mapsto \langle u_0(z), u_{-2}(z), u_{-4}(z), ...\rangle,$$

is $\mathcal{L}$-analytic in Ω, thus the equations

$$\overline{\partial} u_{-2k} + \partial u_{-2k-2} = 0, \tag{34}$$

are satisfied for all $k \geq 0$. Moreover, the hypothesis (31) and the sufficiency part of Theorem 2.2 yields that they extend continuously to $\varGamma$ and

$$u_{-2k}|_\varGamma = g_{-2k}, \quad k \geq 0. \tag{35}$$

Step 2: The construction of positive even modes u_{2n} for $n \geq 1$.

All of the positive even Fourier modes are constructed by conjugation:

$$u_{2k} := \overline{u_{-2k}}, \quad k \geq 1. \tag{36}$$

By conjugating (34) we note that the positive even Fourier modes also satisfy

$$\overline{\partial} u_{2k+2} + \partial u_{2k} = 0, \quad k \geq 0. \tag{37}$$

Moreover, they extend continuously to $\varGamma$ and

$$u_{2k}|_\varGamma = \overline{u_{-2k}}|_\varGamma = \overline{g_{-2k}} = g_{2k}, \quad k \geq 1. \tag{38}$$

Thus, as a summary, we have shown that

$$\overline{\partial} u_{2k} + \partial u_{2k-2} = 0, \quad \forall k \in \mathbb{Z}, \tag{39}$$

$$u_{2k}|_\varGamma = g_{2k}, \quad \forall k \in \mathbb{Z}. \tag{40}$$

Step 3: The construction of modes u_{-1} and u_1.

Let $\psi \in \Psi_g$ as in (30). We define

$$u_{-1} := \psi, \quad \text{and} \quad u_1 := \overline{\psi}. \tag{41}$$

Since g is real valued, we have

$$u_1|_\Gamma = \overline{g_{-1}} = g_1. \tag{42}$$

Step 4: The construction of negative odd modes u_{2n-1} for $n \leq -1$.

Use the Bukhgeim-Cauchy Integral formula (13) to construct the other odd negative Fourier modes:

$$\langle u_{-3}(z), u_{-5}(z), \cdots \rangle := \mathcal{B}\mathbf{g}^{odd}(z), \quad z \in \Omega. \tag{43}$$

By Theorem 2.1, the sequence valued map

$$z \mapsto \langle u_{-3}(z), u_{-5}(z), u_{-7}(z), ..., \rangle,$$

is $\mathcal{L}$-analytic in Ω, thus the equations

$$\overline{\partial} u_{2k-1} + \partial u_{2k-3} = 0, \tag{44}$$

are satisfied for all $k \leq -1$. Moreover, the hypothesis (32) and the sufficiency part of Theorem 2.2 yields that they extend continuously to Γ and

$$u_{2k-1}|_\Gamma = g_{2k-1}, \quad \forall k \leq -1. \tag{45}$$

Step 5: The construction of positive odd modes u_{2n+1} for $n \geq 1$.

All of the positive odd Fourier modes are constructed by conjugation:

$$u_{2k+3} := \overline{u_{-(2k+3)}}, \quad k \geq 0. \tag{46}$$

By conjugating (44) we note that the positive odd Fourier modes also satisfy

$$\overline{\partial} u_{2k+3} + \partial u_{2k+1} = 0, \quad \forall k \geq 1. \tag{47}$$

Moreover, they extend continuously to Γ and

$$u_{2k+3}|_\Gamma = \overline{u_{-(2k+3)}}|_\Gamma = \overline{g_{-(2k+3)}} = g_{2k+3}, \quad k \geq 0. \tag{48}$$

Step 6: The construction of the tensor field $\mathbf{F}_\psi$ whose X-ray data is g.

We define the 2-tensor field

$$\mathbf{F}_\psi := \begin{pmatrix} f_0 + 2\,\mathbb{R}\mathrm{e}\, f_2 & 2\,\mathbb{I}\mathrm{m}\, f_2 \\ 2\,\mathbb{I}\mathrm{m}\, f_2 & f_0 - 2\,\mathbb{R}\mathrm{e}\, f_2 \end{pmatrix}, \tag{49}$$

where

$$f_0 = 2\,\mathbb{R}\mathrm{e}(\partial\psi), \text{ and } f_2 = \overline{\partial}\psi + \partial u_{-3}. \tag{50}$$

In order to show $g|_{\Gamma_+} = X\mathbf{F}_\psi$ with $\mathbf{F}_\psi$ as in (49), we define the real valued function u via its Fourier modes

$$\begin{aligned} u(z,\theta) := {} & u_0(z) + \psi(z)e^{-i\varphi} + \overline{\psi}(z)e^{i\varphi} \\ & + \sum_{n=2}^{\infty} u_{-n}(z)e^{-in\varphi} + \sum_{n=2}^{\infty} u_n(z)e^{in\varphi}, \end{aligned} \tag{51}$$

and check that it has the trace g on $\varGamma$ and satisfies the transport equation (20).

Since $g \in C^{\alpha}\left(\varGamma; C^{1,\alpha}(\mathbf{S}^1)\right) \cap C(\varGamma; C^{2,\alpha}(\mathbf{S}^1))$, we use [30, Corollary 4.1] and [30, Proposition 4.1 (iii)] to conclude that u defined in (51) belongs to $C^{1,\alpha}(\Omega \times \mathbf{S}^1) \cap C^{\alpha}(\overline{\Omega} \times \mathbf{S}^1)$. In particular $u(\cdot, \theta)$ for $\theta = (\cos\varphi, \sin\varphi)$ extends to the boundary and its trace satisfies

$$\begin{aligned}
u(\cdot,\theta)|_{\varGamma} &= \left. \left(u_0 + \psi e^{-i\varphi} + \overline{\psi} e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} u_n e^{in\varphi} \right) \right|_{\varGamma} \\
&= u_0|_{\varGamma} + \psi|_{\varGamma}\, e^{-i\varphi} + \overline{\psi}|_{\varGamma}\, e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n}|_{\varGamma} e^{-in\varphi} + \sum_{n=2}^{\infty} u_n|_{\varGamma} e^{in\varphi} \\
&= g_0 + g_{-1} e^{-i\varphi} + g_1 e^{i\varphi} + \sum_{n=2}^{\infty} g_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} g_n e^{in\varphi} \\
&= g(\cdot, \theta),
\end{aligned}$$

where in the third equality above we used (40), (45),(48), (42) and definition of $\psi \in \Psi_g$ in (30).

Since $u \in C^{1,\alpha}(\Omega \times \mathbf{S}^1) \cap C^{\alpha}(\overline{\Omega} \times \mathbf{S}^1)$, the following calculation is also justified:

$$\begin{aligned}
\theta \cdot \nabla u = e^{-i\varphi}\overline{\partial} u_0 + e^{i\varphi} \partial u_0 + + e^{-2i\varphi} \overline{\partial} \psi + \overline{\partial \psi} + \partial \psi + e^{2i\varphi} \partial \overline{\psi} \\
+ \sum_{n=2}^{\infty} \overline{\partial} u_{-n} e^{-i(n+1)\varphi} + \sum_{n=2}^{\infty} \partial u_{-n} e^{-i(n-1)\varphi} \\
+ \sum_{n=2}^{\infty} \overline{\partial} u_n e^{i(n-1)\varphi} + \sum_{n=2}^{\infty} \partial u_n e^{i(n+1)\varphi}.
\end{aligned}$$

Rearranging the modes in the above equation yields

$$\begin{aligned}
\theta \cdot \nabla u = e^{-2i\varphi} (\overline{\partial} \psi + \partial u_{-3}) + e^{2i\varphi} (\partial \overline{\psi} + \overline{\partial} u_3) + \overline{\partial \psi} + \partial \psi \\
+ e^{-i\varphi} (\overline{\partial} u_0 + \partial u_{-2}) + e^{i\varphi} (\partial u_0 + \overline{\partial} u_2) \\
+ \sum_{n=1}^{\infty} (\overline{\partial} u_{-n} + \partial u_{-n-2}) e^{-i(n+1)\varphi} + \sum_{n=1}^{\infty} (\overline{\partial} u_{n+2} + \partial u_n) e^{i(n+1)\varphi}.
\end{aligned}$$

Using (39), (44), and (47) simplifies the above equation

$$\theta \cdot \nabla u = e^{-2i\varphi} (\overline{\partial} \psi + \partial u_{-3}) + e^{2i\varphi} (\partial \overline{\psi} + \overline{\partial} u_3) + \overline{\partial \psi} + \partial \psi.$$

Now using (50), we conclude (20).

$$\theta \cdot \nabla u = e^{-2i\varphi} f_2 + e^{2i\varphi} \overline{f_2} + f_0 = \langle \mathbf{F}_{\psi} \theta, \theta \rangle.$$

$\square$

As the source is supported inside, there are no incoming fluxes: hence the trace of a solution u of (20) on $\varGamma_-$ is zero. We give next a range condition only in terms of g on $\varGamma_+$, where $g := u|_{\varGamma\times\mathbf{S}^1}$. More precisely, let $\tilde{u}$ be the solution of the boundary value problem

$$\begin{aligned} &\theta\cdot\nabla\tilde{u}(x,\theta) = \langle \mathbf{F}(x)\theta,\theta\rangle, \quad x\in\Omega, \\ &\tilde{u}(z,\theta) = -\frac{1}{2}g|_{\varGamma_+}(z,-\theta), \quad (z,\theta)\in\varGamma_-. \end{aligned} \tag{52}$$

Then one can see that

$$\tilde{u}|_{\varGamma_+} = \frac{1}{2}g|_{\varGamma_+}, \tag{53}$$

and therefore $\tilde{u}|_{\varGamma\times\mathbf{S}^1}$ is an odd function of θ. This shows that we can work with the following odd extension:

$$\tilde{g}(z,\theta) := \frac{g(z,\theta)-g(z,-\theta)}{2}, \quad (z,\theta)\in(\varGamma\times\mathbf{S}^1)\backslash\varGamma_0, \tag{54}$$

and $\tilde{g}=0$ on $\varGamma_0$. Note that $\tilde{g}$ is the trace of $\tilde{u}$ on $\varGamma\times\mathbf{S}^1$.

The range characterization can be given now in terms of the odd Fourier modes of $\tilde{g}$, namely in terms of

$$\tilde{\mathbf{g}} := \langle \tilde{g}_{-3}, \tilde{g}_{-5}, \tilde{g}_{-7}, \ldots\rangle. \tag{55}$$

Corollary 3.1. *Let $\alpha > 1/2$.*

(i) Let $\mathbf{F}\in C_0^{1,\alpha}(\Omega;\mathbb{R}^{2\times2})$, $\tilde{u}$ be the solution of (52) *and $\tilde{\mathbf{g}}$ as in* (55)*. Then $\tilde{\mathbf{g}}\in l^{1,1}_{\infty}(\Gamma)\cap C^{\alpha}(\Gamma;l_1)$ and*

$$[I+i\mathcal{H}]\tilde{\mathbf{g}} = 0, \tag{56}$$

where the operator $\mathcal{H}$ is the Hilbert transform in (16)*.*

(ii) Let $g\in C^{\alpha}(\varGamma;C^{1,\alpha}(\mathbf{S}^1))\cap C(\varGamma;C^{2,\alpha}(\mathbf{S}^1))$ be real valued with $g|_{\varGamma_-\cup\varGamma_0}=0$. Let $\tilde{g}$ be its odd extension as in (54) *and the corresponding $\tilde{\mathbf{g}}$ as in* (55)*. If $\tilde{\mathbf{g}}$ satisfies* (56)*, then there exists a real valued symmetric 2-tensor $\mathbf{F}\in C(\Omega;\mathbb{R}^{2\times2})$, such that $g|_{\varGamma_+}=X\mathbf{F}$. Moreover for each $\psi\in\Psi_g$ in* (30)*, there is a unique real valued symmetric 2-tensor $\mathbf{F}_\psi$ such that $g|_{\varGamma_+}=X\mathbf{F}_\psi$.*

4. The attenuated case

The results in this section need further regularity on the boundary of the domain. We assume that Γ is $C^{2,\alpha}$ for $\alpha>\frac{1}{2}$. We also assume an attenuation $a\in C^{2,\alpha}(\overline{\Omega})$, $\alpha>1/2$ with

$$\min_{\overline{\Omega}} a > 0.$$

We establish necessary and sufficient conditions for a sufficiently smooth function g on $\varGamma\times\mathbf{S}^1$ to be the attenuated X-ray data, with attenuation a, of some sufficiently smooth real symmetric 2-tensor, i.e. g is the trace on $\varGamma \times \mathbf{S}^1$ of some solution u of

$$\theta \cdot \nabla u(x,\theta) + a(x)u(x,\theta) = \langle \mathbf{F}(x)\theta, \theta\rangle, \quad (x,\theta) \in \varGamma \times \mathbf{S}^1. \tag{57}$$

Different from the 1-tensor case in [31] (where there is uniqueness), in the 2-tensor case there is non-uniqueness: see the class of function in (82).

As in [30] we start by the reduction to the non-attenuated case via the special integrating factor e^{-h}, where h is explicitly defined in terms of a by

$$h(z,\theta) := Da(z,\theta) - \frac{1}{2}\left(I - iH\right) Ra(z\cdot\theta^{\perp}, \theta^{\perp}), \tag{58}$$

where $\theta^{\perp}$ is orthogonal to θ, $Da(z,\theta) = \displaystyle\int_0^{\infty} a(z+t\theta)dt$ is the divergent beam transform of the attenuation a, $Ra(s,\theta^{\perp}) = \displaystyle\int_{-\infty}^{\infty} a\left(s\theta^{\perp} + t\theta\right) dt$ is the Radon transform of the attenuation a, and the classical Hilbert transform $Hh(s) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{h(t)}{s-t}dt$ is taken in the first variable and evaluated at $s = z\cdot\theta^{\perp}$. The function h was first considered in the work of Natterer [21]; see also [8], and [6] for elegant arguments that show how h extends from $\mathbf{S}^1$ inside the disk as an analytic map.

The lemma 4.1 and lemma 4.2 below were proven in [31] for a vanishing at the boundary. Under the smoother regularity assumption on Γ, a need not to vanish at the boundary. This is because the map $\overline{\Omega} \times \mathbf{S}^1 \ni (z,\theta) \mapsto \tau_+(z,\theta)$ is in $C^{2,\alpha}(\overline{\Omega} \times \mathbf{S}^1)$, where τ_+ denote the distance from z to the boundary in the direction $+\theta$.

Lemma 4.1. [31, Lemma 4.1] *Assume Ω is $C^{2,\alpha}$ convex domain and $a \in C^{p,\alpha}(\overline{\Omega})$, $p = 1,2$, $\alpha > 1/2$, and h defined in (58). Then $h \in C^{p,\alpha}(\overline{\Omega} \times \mathbf{S}^1)$ and the following hold*

(i) h satisfies

$$\theta \cdot \nabla h(z,\theta) = -a(z),\ (z,\theta) \in \Omega \times \mathbf{S}^1. \tag{59}$$

(ii) h has vanishing negative Fourier modes yielding the expansions

$$e^{-h(z,\theta)} := \sum_{k=0}^{\infty} \alpha_k(z) e^{ik\varphi}, \quad e^{h(z,\theta)} := \sum_{k=0}^{\infty} \beta_k(z) e^{ik\varphi},\ (z,\theta) \in \overline{\Omega} \times \mathbf{S}^1, \tag{60}$$

with

(iii)

$$z \mapsto \langle \alpha_1(z), \alpha_2(z), \alpha_3(z), ..., \rangle \in C^{p,\alpha}(\Omega; l_1) \cap C(\overline{\Omega}; l_1), \tag{61}$$

$$z \mapsto \langle \beta_1(z), \beta_2(z), \beta_3(z), ..., \rangle \in C^{p,\alpha}(\Omega; l_1) \cap C(\overline{\Omega}; l_1). \tag{62}$$

(iv) For any $z \in \Omega$

$$\overline{\partial}\beta_0(z) = 0, \tag{63}$$

$$\overline{\partial}\beta_1(z) = -a(z)\beta_0(z), \tag{64}$$

$$\overline{\partial}\beta_{k+2}(z) + \partial\beta_k(z) + a(z)\beta_{k+1}(z) = 0,\ k \geq 0. \tag{65}$$

(v) For any $z \in \Omega$

$$\overline{\partial}\alpha_0(z) = 0, \tag{66}$$

$$\overline{\partial}\alpha_1(z) = a(z)\alpha_0(z), \tag{67}$$

$$\overline{\partial}\alpha_{k+2}(z) + \partial\alpha_k(z) + a(z)\alpha_{k+1}(z) = 0,\ k \geq 0. \tag{68}$$

(vi) The Fourier modes $\alpha_k, \beta_k, k \geq 0$ satisfy

$$\alpha_0\beta_0 = 1, \quad \sum_{m=0}^{k} \alpha_m\beta_{k-m} = 0, k \geq 1. \tag{69}$$

From (59) it is easy to see that u solves (57) if and only if $v := e^{-h}u$ solves

$$\theta \cdot \nabla v(z,\theta) = \langle F(z)\theta, \theta\rangle e^{-h(z,\theta)}. \tag{70}$$

If $u(z,\theta) = \sum_{n=-\infty}^{\infty} u_n(z)e^{in\varphi}$ solves (57), then its Fourier modes satisfy

$$\overline{\partial}u_1(z) + \partial u_{-1}(z) + a(z)u_0(z) = f_0(z), \tag{71}$$

$$\overline{\partial}u_0(z) + \partial u_{-2}(z) + a(z)u_{-1}(z) = 0, \tag{72}$$

$$\overline{\partial}u_{-1}(z) + \partial u_{-3}(z) + a(z)u_{-2}(z) = f_2(z), \tag{73}$$

$$\overline{\partial}u_n(z) + \partial u_{n-2}(z) + a(z)u_{n-1}(z) = 0, \quad n \leq -2, \tag{74}$$

where f_0, f_2 as defined in (19).

Also, if $v := e^{-h}u = \sum_{n=-\infty}^{\infty} v_n(z)e^{in\varphi}$ solves (70), then its Fourier modes satisfy

$$\begin{aligned}
&\overline{\partial}v_1(z) + \partial v_{-1}(z) = \alpha_0(z)f_0(z) + \alpha_2(z)f_2(z),\\
&\overline{\partial}v_0(z) + \partial v_{-2}(z) = \alpha_1(z)f_2(z),\\
&\overline{\partial}v_{-1}(z) + \partial v_{-3}(z) = \alpha_0(z)f_2(z),\\
&\overline{\partial}v_n(z) + \partial v_{n-2}(z) = 0, \quad n \leq -2,
\end{aligned} \tag{75}$$

where α_0, α_1 and α_2 are the Fourier modes in (60), and f_0, f_2 as defined in (19).

The following result shows that the equivalence between (74) and (75) is intrinsic to negative Fourier modes only.

Lemma 4.2. [31, Lemma 4.2] *Assume* $a \in C^{1,\alpha}(\overline{\Omega}), \alpha > 1/2$.

(i) Let $\mathbf{v} = \langle v_{-2}, v_{-3}, ...\rangle \in C^1(\Omega, l_1)$ *satisfy* (75)*, and* $\mathbf{u} = \langle u_{-2}, u_{-3}, ...\rangle$ *be defined componentwise by the convolution*

$$u_n := \sum_{j=0}^{\infty} \beta_j v_{n-j}, \quad n \leq -2, \tag{76}$$

where β_j*'s are the Fourier modes in* (60)*. Then* $\mathbf{u}$ *solves* (74) *in* Ω*.*

(ii) Conversely, let $\mathbf{u} = \langle u_{-2}, u_{-3}, ...\rangle \in C^1(\Omega, l_1)$ *satisfy* (74)*, and* $\mathbf{v} = \langle v_{-2}, v_{-3}, ...\rangle$ *be defined componentwise by the convolution*

$$v_n := \sum_{j=0}^{\infty} \alpha_j u_{n-j}, \quad n \leq -2, \tag{77}$$

where α_j*'s are the Fourier modes in* (60)*. Then* $\mathbf{v}$ *solves* (75) *in* Ω*.*

The operators ∂, $\overline{\partial}$ in (9) can be rewritten in terms of the derivative in tangential direction ∂_τ and derivative in normal direction ∂_n,

$$\partial_n = \cos\eta\partial_{x_1} + \sin\eta\partial_{x_2},$$
$$\partial_\tau = -\sin\eta\partial_{x_1} + \cos\eta\partial_{x_2},$$

where η is the angle made by the normal to the boundary with x_1 direction (Since the boundary $\varGamma$ is known, η is a known function on the boundary). In these coordinates

$$\partial = \frac{e^{-i\eta}}{2}(\partial_n - i\partial_\tau), \quad \overline{\partial} = \frac{e^{i\eta}}{2}(\partial_n + i\partial_\tau). \tag{78}$$

Next we characterize the attenuated X-ray data g in terms of its Fourier modes g_0, g_{-1} and the negative index modes $\gamma_{-2}, \gamma_{-3}, \gamma_{-4}...$ of

$$e^{-h(\zeta,\theta)}g(\zeta,\theta) = \sum_{k=-\infty}^{\infty} \gamma_k(\zeta)e^{ik\varphi}, \quad \zeta \in \varGamma. \tag{79}$$

To simplify the statement, let

$$\mathbf{g}_h := \langle \gamma_{-2}, \gamma_{-3}, \gamma_{-4}...\rangle, \tag{80}$$

and from the negative even, respectively, negative odd Fourier modes, we built the sequences

$$\mathbf{g}_h^{even} = \langle \gamma_{-2}, \gamma_{-4}, ...\rangle, \quad \text{and} \quad \mathbf{g}_h^{odd} = \langle \gamma_{-3}, \gamma_{-5}, ...\rangle. \tag{81}$$

Note that γ_{-1} is not included in the $\mathbf{g}_h^{odd}$ definition. As before we construct simultaneously the right hand side of the transport equation (57) together with the solution u. Construction of u is via its Fourier modes. We first construct the negative modes and then the positive modes are constructed by conjugation. Apart from zeroth mode u_0 and negative one mode u_{-1}, all Fourier modes are constructed uniquely from the data $\mathbf{g}_h^{even}$, $\mathbf{g}_h^{odd}$. The mode u_0 will be chosen arbitrarily from the class Ψ_g^a with prescribed trace and gradient on the boundary $\varGamma$ defined as

$$\begin{aligned}(82)\quad \Psi_g^a := \Bigg\{ &\psi \in C^2(\overline{\Omega};\mathbb{R}) : \psi|_{\varGamma} = g_0, \\ &\partial_n \psi|_{\varGamma} = -2\,\mathbb{R}\mathrm{e}\, e^{-i\eta} \left(\partial \sum_{j=0}^{\infty} \beta_j (\mathcal{B}\mathbf{g}_h)_{-2-j} \Bigg|_{\varGamma} + a|_{\varGamma}\, g_{-1} \right) \Bigg\},\end{aligned}$$

where $\mathcal{B}$ be the Bukhgeim-Cauchy operator in (13), β_j's are the Fourier modes in (60) and $\mathbf{g}_h$ in (80). The mode u_{-1} is define in terms of u_0, see (99).

Recall the Hilbert transform $\mathcal{H}$ in (16).

Theorem 4.1 (Range characterization in the attenuated case)**.** *Let $a \in C^{2,\alpha}(\overline{\Omega})$, $\alpha > 1/2$ with $\min_{\overline{\Omega}} a > 0$.*

(i) Let $\mathbf{F} \in C_0^{1,\alpha}(\Omega;\mathbb{R}^{2\times 2})$. For $g := \begin{cases} X_a\mathbf{F}(x,\theta), & (x,\theta) \in \varGamma_+, \\ 0, & (x,\theta) \in \varGamma_- \cup \varGamma_0, \end{cases}$ consider the corresponding sequences $\mathbf{g}_h^{even}, \mathbf{g}_h^{odd}$ as in (81)*. Then $\mathbf{g}_h^{even}, \mathbf{g}_h^{odd} \in l_{\infty}^{1,1}(\Gamma) \cap C^{\alpha}(\Gamma; l_1)$ satisfy*

$$(83)\qquad [I + i\mathcal{H}]\mathbf{g}_h^{even} = 0, \quad [I + i\mathcal{H}]\mathbf{g}_h^{odd} = 0, \quad \textit{and}$$

$$(84)\qquad \partial_\tau g_0 = -2\,\mathbb{I}\mathrm{m}\, e^{-i\eta} \left(\partial \sum_{j=0}^{\infty} \beta_j (\mathcal{B}\mathbf{g}_h)_{-2-j} \Bigg|_{\varGamma} + a|_{\varGamma}\, g_{-1} \right),$$

where $\mathcal{H}$ is the Hilbert transform in (16)*, $\mathcal{B}$ is the Bukhgeim-Cauchy operator in* (13)*, β_j's are the Fourier modes in* (60) *and $\mathbf{g}_h$ in* (80)*.*

(ii) Let $g \in C^{\alpha}(\varGamma; C^{1,\alpha}(\mathbf{S}^1)) \cap C(\varGamma; C^{2,\alpha}(\mathbf{S}^1))$ be real valued with $g|_{\varGamma_- \cup \varGamma_0} = 0$. If the corresponding sequences $\mathbf{g}_h^{even}, \mathbf{g}_h^{odd} \in Y_\alpha$ satisfying (83) *and* (84) *then there exists a symmetric 2-tensor $\mathbf{F} \in C(\Omega;\mathbb{R}^{2\times 2})$, such that $g|_{\varGamma_+} = X_a\mathbf{F}$. Moreover for each $\psi \in \Psi_g^a$ in* (82)*, there is a unique real valued symmetric 2-tensor $\mathbf{F}_\psi$ such that $g|_{\varGamma_+} = X_a\mathbf{F}_\psi$.*

Proof. (i) **Necessity**

Let $\mathbf{F} \in C_0^{1,\alpha}(\Omega;\mathbb{R}^{2\times 2})$. Since $\mathbf{F}$ is compactly supported inside Ω, for any point at the boundary there is a cone of lines which do not meet the support. Thus $g \equiv 0$ in the neighborhood of the variety $\varGamma_0$ which yields $g \in C^{1,\alpha}(\varGamma \times \mathbf{S}^1)$. Moreover, g is the trace on $\varGamma \times \mathbf{S}^1$ of a solution $u \in C^{1,\alpha}(\overline{\Omega} \times \mathbf{S}^1)$. By [30, Proposition 4.1] $\mathbf{g}_h^{even}, \mathbf{g}_h^{odd} \in l_{\infty}^{1,1}(\Gamma) \cap C^{\alpha}(\Gamma; l_1)$.

Let $v := e^{-h}u = \sum_{n=-\infty}^{\infty} v_n(z)e^{in\varphi}$, then the negative Fourier modes of v satisfy (75). In particular its negative odd subsequence $\langle v_{-3}, v_{-5}, \ldots\rangle$ and negative even subsequence $\langle v_{-2}, v_{-4}, \ldots\rangle$ are $\mathcal{L}$-analytic with traces $\mathbf{g}_h^{odd}$

respectively $\mathbf{g}_h^{even}$. The necessity part of Theorem 2.2 yields (83):

$$[I+i\mathcal{H}]\mathbf{g}_h^{odd}=0, \quad [I+i\mathcal{H}]\mathbf{g}_h^{even}=0.$$

If u solves (57), then its Fourier modes satisfy (71), (72), (73), and (74). The negative Fourier modes of u and v are related by

$$u_n=\sum_{j=0}^{\infty}\beta_j v_{n-j}, \quad n\leq 0, \tag{85}$$

where β_j's are the Fourier modes in (60). The restriction of (72) to the boundary yields

$$\overline{\partial}u_0|_{\varGamma}=-\partial u_{-2}|_{\varGamma}-(au_{-1})|_{\varGamma}.$$

Expressing $\overline{\partial}$ in the above equation in terms of ∂_τ and ∂_n as in (78) yields

$$\frac{e^{i\eta}}{2}(\partial_n+i\partial_\tau)u_0|_{\varGamma}=-\partial u_{-2}|_{\varGamma}-a|_{\varGamma}g_{-1}.$$

Simplifying the above expression and using $\partial_\tau u_0|_{\varGamma}=\partial_\tau g_0$, yields

$$\partial_n u_0|_{\varGamma}+i\partial_\tau g_0=-2e^{-i\eta}\left(\partial u_{-2}|_{\varGamma}+a|_{\varGamma}g_{-1}\right).$$

The imaginary part of the above equation yields (84). This proves part (i) of the theorem.

(ii) **Sufficiency**

To prove the sufficiency we will construct a real valued symmetric 2-tensor $\mathbf{F}$ in Ω and a real valued function $u\in C^1(\Omega\times\mathbf{S}^1)\cap C(\overline{\Omega}\times\mathbf{S}^1)$ such that $u|_{\varGamma\times\mathbf{S}^1}=g$ and u solves (57) in Ω. The construction of such u is in terms of its Fourier modes in the angular variable and it is done in several steps.

Step 1: The construction of negative modes u_n for $n\leq -2$.

Let $g\in C^\alpha\left(\varGamma;C^{1,\alpha}(\mathbf{S}^1)\right)\cap C(\varGamma;C^{2,\alpha}(\mathbf{S}^1))$ be real valued with $g|_{\varGamma_-\cup\varGamma_0}=0$. Let the corresponding sequences $\mathbf{g}_h^{even},\mathbf{g}_h^{odd}$ as in (81) satisfying (83) and (84). By [30, Proposition 4.1(ii)] and [30, Proposition 5.2(iii)] $\mathbf{g}_h^{even},\mathbf{g}_h^{odd}\in Y_\alpha$. Use the Bukhgeim-Cauchy Integral formula (13) to define the $\mathcal{L}$-analytic maps

$$\mathbf{v}^{even}(z)=\langle v_{-2}(z),v_{-4}(z),...\rangle:=\mathcal{B}\mathbf{g}_h^{even}(z), \quad z\in\Omega, \tag{86}$$

$$\mathbf{v}^{odd}(z)=\langle v_{-3}(z),v_{-5}(z),...\rangle:=\mathcal{B}\mathbf{g}_h^{odd}(z), \quad z\in\Omega. \tag{87}$$

By intertwining let also define

$$\mathbf{v}(z):=\langle v_{-2}(z),v_{-3}(z),...\rangle, \quad z\in\Omega.$$

By Proposition 2.1

$$\mathbf{v}^{even},\mathbf{v}^{odd},\mathbf{v}\in C^{1,\alpha}(\Omega;l_1)\cap C^\alpha(\overline{\Omega};l_1)\cap C^2(\Omega;l_\infty). \tag{88}$$

Moreover, since $\mathbf{g}_h^{even}, \mathbf{g}_h^{odd}$ satisfy the hypothesis (83), by Theorem 2.2 we have

$$\mathbf{v}^{even}|_{\Gamma} = \mathbf{g}_h^{even} \quad \text{and} \quad \mathbf{v}^{odd}|_{\Gamma} = \mathbf{g}_h^{odd}.$$

In particular

$$v_n|_{\Gamma} = \sum_{k=0}^{\infty} (\alpha_k|_{\Gamma})\, g_{n-k}, \quad n \leq -2. \tag{89}$$

For each $n \leq -2$, we use the convolution formula below to construct

$$u_n := \sum_{j=0}^{\infty} \beta_j v_{n-j}. \tag{90}$$

Since $a \in C^{2,\alpha}(\overline{\Omega})$, by (62), the sequence $z \mapsto \langle \beta_0(z), \beta_1(z), \beta_2(z), ...\rangle$ is in $C^{2,\alpha}(\Omega; l_1) \cap C^{\alpha}(\overline{\Omega}; l_1)$. Since convolution preserves l_1, the map is in

$$z \mapsto \langle u_{-2}(z), u_{-3}(z), ...\rangle \in C^{1,\alpha}(\Omega; l_1) \cap C^{\alpha}(\overline{\Omega}; l_1). \tag{91}$$

Moreover, since $\mathbf{v} \in C^2(\Omega; l_\infty)$ as in (88), we also conclude from convolution that

$$z \mapsto \langle u_{-2}(z), u_{-3}(z), ...\rangle \in C^2(\Omega; l_\infty). \tag{92}$$

The property (91) justifies the calculation of traces $u_n|_{\Gamma}$ for each $n \leq -2$:

$$u_n|_{\Gamma} = \sum_{j=0}^{\infty} \beta_j|_{\Gamma} (v_{n-j}|_{\Gamma}).$$

Using (89) in the above equation gives

$$u_n|_{\Gamma} = \sum_{j=0}^{\infty} \beta_j|_{\Gamma} \sum_{k=0}^{\infty} \alpha_k|_{\Gamma} g_{n-j-k}.$$

A change of index $m = j + k$, simplifies the above equation

$$\begin{aligned} u_n|_{\Gamma} &= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \alpha_k \beta_{m-k} g_{n-m}, \\ &= \alpha_0 \beta_0 g_n + \sum_{m=1}^{\infty} \sum_{k=0}^{m} \alpha_k \beta_{m-k} g_{n-m}. \end{aligned}$$

Using Lemma 4.1 (vi) yields

$$u_n|_{\Gamma} = g_n, \quad n \leq -2. \tag{93}$$

From the Lemma 4.2, the constructed u_n in (90) satisfy

$$\overline{\partial} u_n + \partial u_{n-2} + a u_{n-1} = 0, \quad n \leq -2. \tag{94}$$

Step 2: The construction of positive modes u_n for $n \geq 2$.

All of the positive Fourier modes are constructed by conjugation:

$$u_n := \overline{u_{-n}}, \quad n \geq 2. \tag{95}$$

Moreover using (93), the traces $u_n|_\Gamma$ for each $n \geq 2$:

$$u_n|_\Gamma = \overline{u_{-n}}|_\Gamma = \overline{g_{-n}} = g_n, \quad n \geq 2. \tag{96}$$

By conjugating (94) we note that the positive Fourier modes also satisfy

$$\overline{\partial} u_{n+2} + \partial u_n + a u_{n+1} = 0, \quad n \geq 2. \tag{97}$$

Step 3: The construction of modes u_0, u_{-1} and u_1.
Let $\psi \in \Psi_g^a$ as in (82) and define

$$u_0 := \psi, \tag{98}$$

and

$$u_{-1} := \frac{-\overline{\partial}\psi - \partial u_{-2}}{a}, \quad u_1 := \overline{u_{-1}}. \tag{99}$$

By the construction $u_0 \in C^2(\Omega; l_\infty)$ and $u_{-1} \in C^1(\Omega; l_\infty)$, and

$$\overline{\partial} u_0 + \partial u_{-2} + a u_{-1} = 0 \tag{100}$$

is satisfied. Furthermore, by conjugating (100) yields

$$\partial u_0 + \overline{\partial} u_2 + a u_1 = 0. \tag{101}$$

Since $\psi \in \Psi_g^a$, the trace of u_0 satisfies

$$u_0|_\Gamma = g_0. \tag{102}$$

We check next that the trace of u_{-1} is g_{-1}:

$$\begin{aligned} u_{-1}|_\Gamma &= \left.\frac{-\overline{\partial}\psi - \partial u_{-2}}{a}\right|_\Gamma \\ &= -\left.\frac{1}{a}\right|_\Gamma \frac{e^{i\eta}}{2}(\partial_n + i\partial_\tau)\psi|_\Gamma - \left.\frac{1}{a}\right|_\Gamma \partial u_{-2}|_\Gamma \\ &= -\left.\frac{1}{2a}\right|_\Gamma e^{i\eta}\left\{\partial_n\psi|_\Gamma + i\partial_\tau\psi|_\Gamma + 2e^{-i\eta}\partial u_{-2}|_\Gamma\right\} \\ &= g_{-1}, \end{aligned} \tag{103}$$

where the last equality uses (84) and the condition in class (82).

Step 4: The construction of the tensor field $\mathbf{F}_\psi$ whose attenuated X-ray data is g.

We define the 2-tensor

$$\mathbf{F}_\psi := \begin{pmatrix} f_0 + 2\,\mathbb{R}\mathrm{e}\, f_2 & 2\,\mathbb{I}\mathrm{m}\, f_2 \\ 2\,\mathbb{I}\mathrm{m}\, f_2 & f_0 - 2\,\mathbb{R}\mathrm{e}\, f_2 \end{pmatrix}, \tag{104}$$

where

$$f_0 = -2\,\mathbb{R}\mathrm{e}\left(\frac{\overline{\partial}\psi + \partial u_{-2}}{a}\right) + a\psi, \text{ and} \tag{105}$$

$$f_2 = -\overline{\partial}\left(\frac{\overline{\partial}\psi + \partial u_{-2}}{a}\right) + \partial u_{-3} + a u_{-2}. \tag{106}$$

Note that f_2 is well defined as $u_{-2} \in C^2(\Omega; l_\infty)$ from (92).

In order to show $g|_{\Gamma_+} = X_a \mathbf{F}_\psi$ with $\mathbf{F}_\psi$ as in (104), we define the real valued function u via its Fourier modes

$$\begin{aligned} u(z,\theta) := {} & u_0(z) + u_{-1}e^{-i\varphi} + \overline{u_{-1}}(z)e^{i\varphi} \\ & + \sum_{n=2}^{\infty} u_{-n}(z)e^{-in\varphi} + \sum_{n=2}^{\infty} u_n(z)e^{in\varphi}. \end{aligned} \tag{107}$$

We check below that u is well defined, has the trace g on Γ and satisfies the transport equation (57).

For convenience consider the intertwining sequence

$$\mathbf{u}(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), u_{-3}(z), \ldots\rangle, \quad z \in \Omega.$$

Since $\mathbf{u} \in C^{1,\alpha}(\Omega; l_1) \cap C^{\alpha}(\overline{\Omega}; l_1)$, by [30, Proposition 4.1 (iii)] we conclude that u is well defined by (107) and as a function in $C^{1,\alpha}(\Omega \times \mathbf{S}^1) \cap C^{\alpha}(\overline{\Omega} \times \mathbf{S}^1)$. In particular $u(\cdot,\theta)$ for $\theta = (\cos\varphi, \sin\varphi)$ extends to the boundary and its trace satisfies

$$\begin{aligned} u(\cdot,\theta)|_\Gamma &= \left.\left(u_0 + u_{-1}e^{-i\varphi} + \overline{u_{-1}}e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n}e^{-in\varphi} + \sum_{n=2}^{\infty} u_n e^{in\varphi}\right)\right|_\Gamma \\ &= u_0|_\Gamma + u_{-1}|_\Gamma e^{-i\varphi} + \overline{u_{-1}}|_\Gamma e^{i\varphi} + \sum_{n=2}^{\infty}(u_{-n}|_\Gamma)e^{-in\varphi} + \sum_{n=2}^{\infty}(u_n|_\Gamma)e^{in\varphi} \\ &= g_0 + g_{-1}e^{-i\varphi} + g_1 e^{i\varphi} + \sum_{n=2}^{\infty} g_{-n}e^{-in\varphi} + \sum_{n=2}^{\infty} g_n e^{in\varphi} \\ &= g(\cdot,\theta), \end{aligned}$$

where is the third equality we have used (93), (96), (102), and (103).

Since $u \in C^{1,\alpha}(\Omega \times \mathbf{S}^1) \cap C^{\alpha}(\overline{\Omega} \times \mathbf{S}^1)$, the following calculation is also justified:

$$\begin{aligned}\theta \cdot \nabla u + au = {} & e^{-i\varphi}\overline{\partial} u_0 + e^{i\varphi}\partial u_0 + e^{-2i\varphi}\overline{\partial} u_{-1} + \overline{\partial} u_1 + \partial u_{-1} + e^{2i\varphi}\partial u_1 \\ & + \sum_{n=2}^{\infty} \overline{\partial} u_{-n} e^{-i(n+1)\varphi} + \sum_{n=2}^{\infty} \partial u_{-n} e^{-i(n-1)\varphi} \\ & + \sum_{n=2}^{\infty} \overline{\partial} u_n e^{i(n-1)\varphi} + \sum_{n=2}^{\infty} \partial u_n e^{i(n+1)\varphi} \\ & + au_0 + au_{-1}e^{-i\varphi} + au_1 e^{i\varphi} + \sum_{n=2}^{\infty} au_{-n}e^{-in\varphi} + \sum_{n=2}^{\infty} au_n e^{in\varphi}.\end{aligned}$$

Rearranging the modes in the above equation yields

$$\begin{aligned}\theta \cdot \nabla u + au = {} & e^{-2i\varphi}(\overline{\partial} u_{-1} + \partial u_{-3} + au_{-2}) + e^{2i\varphi}(\partial u_1 + \overline{\partial} u_3 + au_2) \\ & + e^{-i\varphi}(\overline{\partial} u_0 + \partial u_{-2} + au_{-1}) + e^{i\varphi}(\partial u_0 + \overline{\partial} u_2 + au_1) \\ & + \overline{\partial} u_1 + \partial u_{-1} + au_0 + \sum_{n=2}^{\infty}(\overline{\partial} u_{n+2} + \partial u_n + au_{n+1})e^{i(n+1)\varphi} \\ & + \sum_{n=2}^{\infty}(\overline{\partial} u_{-n} + \partial u_{-n-2} + au_{-n-1})e^{-i(n+1)\varphi}.\end{aligned}$$

Using (94), (97), (100) and (101) simplifies the above equation

$$\begin{aligned}\theta \cdot \nabla u + au = {} & e^{-2i\varphi}(\overline{\partial} u_{-1} + \partial u_{-3} + au_{-2}) + e^{2i\varphi}(\partial u_1 + \overline{\partial} u_3 + au_2) \\ & + \overline{\partial} u_1 + \partial u_{-1} + au_0.\end{aligned}$$

Now using (105) and (106), we conclude (57)

$$\theta \cdot \nabla u + au = e^{-2i\varphi} f_2 + e^{2i\varphi}\overline{f_2} + f_0 = \langle \mathbf{F}_\psi \theta, \theta \rangle.$$

□

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